h-PRINCIPLE IN SYMPLECTIC GEOMETRY

WENYUAN LI

ABSTRACT. These are some notes on h-principle and its applications in symplectic and contact geometry. The main source will be Eliashberg and Mishachev's *Introduction to the h-Principle*. For some h-principles for closed contact manifolds the source is Geige's *An Introduction to Contact Topology*. For the h-principle for loose Legendrians, the source is Murphy's *Loose Legendrian Embeddings in High Dimensional Contact Manifolds*.

CONTENTS

1. JET BUNDLES

1.1. **Definition of Jet Bundles.** Recall that for a smooth manifold M, to encode the information of the differentials of smooth functions, we can construct the cotangent bundle.

Namely let $\mathcal{C}_x^\infty$ be the stalk of the smooth functions at $x \in M$ and $\mathfrak{m}_x \subset \mathcal{C}_x^\infty$ be the unique maximal ideal (consisting of functions that vanish at x). Then $T^*M = \bigcup_{x\in M} T_x^*M$ where $T_x^*M = \mathfrak{m}_x/\mathfrak{m}_x^2$. If in addition we would like to encode the information of the evaluation of smooth functions, we can consider $J^1(M) = \bigcup_{x\in M} J_x^1(M)$ where $J_x^1(M) = \mathcal{C}_x^\infty/\mathfrak{m}_x^2$. In general this leads us to the notion of jet bundles.

Definition 1.1. *Given a smooth fibration $p : X \to V$, the bundle of sections is $X^\infty = \bigcup_{v\in V} X_v^\infty$, where X_v^∞ is the stalk of smooth sections at $v \in V$. Let $\mathcal{R}_v^r$ be the equivalent relation on X_v^∞ such that $s_1 \sim_{\mathcal{R}_v^r} s_2$ if for any (germ of) smooth function $f \in \mathcal{C}_v^\infty$*

$$d^r(f\circ s_1)_x = d^r(f\circ s_2).$$

The r-jet bundle is $X^{(r)} = \bigcup_{v\in V} X_v^{(r)}$, where $X_v^{(r)} = X_v^\infty/\mathcal{R}_v^{r+1}$.

If $X = V\times W$ and $p : X \to V$ be the canonical projection to the first factor. Then we may write the r-jet space of maps $V \to W$ as $X^{(r)} = J^r(V,W)$.

Note that by definition there exists natural projection $p_{r-1}^r : X^{(r)} \to X^{(r-1)}$. We will also write $p_s^r = p_s^{s+1}\circ p_{s+1}^{s+2}\circ ...\circ p_{r-1}^r$. Since the equivalence relation $\mathcal{R}_v^r$ is determined up to an additive constant, this is only an affine bundle, not a vector bundle (only when there is a canonically chosen zero section can we get a vector bundle structure).

Definition 1.2. *For a section $F : V \to X^{(r)}$, the underlying section is $\mathrm{bs}F = p_0^r\circ F : V \to X$. F is a holonomic section if $F = J^r\mathrm{bs}F$ (or $F = d^r\mathrm{bs}F$) where $(J^rf)_v$ (or $(d^rf)_v$) is the equivalent class of f in $X_v^{(r)} = X_v^\infty/\mathcal{R}_v^{r+1}$. The space of holonomic sections is $\mathrm{Hol}\,X^{(r)}$.*

For the notion of higher differentials J^rf (or d^rf), a reference is Warner's *Introduction to Differentiable Manifolds and Lie Groups* Section 1.1.

Proposition 1.1 (Holonomic splitting)**.** *Given any holonomic section $F : V \to X^{(r)}$, any open ball $D \subset V$, there is a local trivialization*

$$P_F : D\times\mathbb{R}^{q(n+r)!/n!r!} \to X^{(r)}$$

such that $P_F(v,0) = F(v)$, and $v \mapsto P_F(v,z)$ is holonomic for any $z \in \mathbb{R}^{q(n+r)!/n!r!}$.

Proof. Identify $\mathbb{R}^{q(n+r)!/n!r!}$ with the space $P^r(\mathbb{R}^n,\mathbb{R}^q)$ of degree r polynomials $\mathbb{R}^n \to \mathbb{R}^q$. Then define the trivialization to be

$$P_F : D\times P^r(\mathbb{R}^n,\mathbb{R}^q) \to X^{(r)};\ \ P_F(v,z) = F(v) + J^rz(v).$$

This because of dimension reason defines a fiberwise isomorphism and for any z, $P_F(v,z)$ is obvioulу holonomic. □

1.2. **Thom Transversality Theorem.** In differential topology, the notion of transversality plays an important role, allowing us to perturb the smooth map in order to get good intersections. The basic technique used in proving transversality is the Sard theorem.

Theorem 1.2 (Sard Theorem)**.** *Let $f : M \to N$ be a smooth map. Then the subset of critical values*

$$\mathrm{Critv}(f) = f(\mathrm{Crit}(f)),\ \mathrm{Crit}(f) = \{x\in M | \mathrm{rank}(df)_x \neq \dim N\}$$

has Lebesgue measure 0.

Corollary 1.3. *Let $f : M \to N$ be a smooth map. If $\dim M < \dim N$ then $f(M)$ has measure zero.*

Now we are ready to prove Thom transversality theorem, which shows that a generic smooth map can be transversal to a manifold up to arbitrary high differentials.

Theorem 1.4 (Thom Transversality Theorem). *Let $X \to V$ be a smooth fibration and $\Sigma \subset X^{(r)}$ a submanifold. Then a generic holonomic section $F : V \to X^{(r)}$ is transverse to Σ with respect to the C^1-topology in $\mathrm{Hol}X^{(r)} \subset \mathrm{Sec}X^{(r)}$.*

Fix any holonomic section $F : V \to X$. By holonomic splitting we know F defines a local trivialization $P_F : D \times P^r(\mathbb{R}^n, \mathbb{R}^q) \to X^{(r)}$. Write $(v_i)_{i\le n}$ the local coordinates of V and $x_j^{\boldsymbol{\alpha}}$ the local coordinates of $X^{(r)}$ determined by this trivialization. Given a polynomial $z \in P(\mathbb{R}^n, \mathbb{R}^q)$, the tangent map is now given by

$$(dz)_v : T_vV \to T_{f(v)}X^{(r)}; (dz)_v\Big(\frac{\partial}{\partial v_i}\Big) = \frac{\partial}{\partial v_i} + \sum_{|\boldsymbol{\alpha}|\le r} \frac{\partial}{\partial v_i}\frac{\partial^{|\boldsymbol{\alpha}|} z_j}{\partial v^{\boldsymbol{\alpha}}}\Big(\frac{\partial}{\partial x_j^{\boldsymbol{\alpha}}}\Big).$$

Proof of Theorem 1.4. We fix an n-disc $D \subset V$ and show that a generic holonomic section $F : D \to X^{(r)}$ is transverse to Σ. Then since transversality is a local property and global sections can be glued from countably many local sections, the proof will be completed.

First it is easy to show that the space of such holonomic sections is open in $\mathrm{Hol}_D X^{(r)}$ with respect to the C^1-topology in $\mathrm{Sec}_D X^{(r)}$, which is the C^{r+1}-topology in $\mathrm{Sec}_D(X)$. Then we have to show that it is dense. Now denote $\dim \Sigma = k$. We consider two different cases.

When $\dim V + \dim \Sigma < \dim X$, we show that the space of sections $F \in \mathrm{Hol}_D X^{(r)}$, $F(V) \cap \Sigma = \emptyset$ is dense. In fact, under the local trivialization P_F, the space of $z \in P^r(\mathbb{R}^n, \mathbb{R}^q)$ such that $z(D) \cap \Sigma \neq \emptyset$ is dense by Sard's theorem since $\dim \Sigma < \dim P^r(\mathbb{R}^n, \mathbb{R}^q)$. This proves the result.

When $\dim V + \dim \Sigma \ge \dim X$, under the local trivialization P_F, the basis of $T_{f(v)}\Sigma$ is $(\nu_i^l|_{i\le q}, \sigma_{\boldsymbol{\alpha}}^l|_{|\boldsymbol{\alpha}|\le r})_{l\le k}$. Then the space spanned by $(dz)_v(T_vV)$ and $T_{f(v)}\Sigma$ is

$$\begin{pmatrix}
1 & 0 & \dots & 0 & \partial_{v_1} z_1 & \dots & \partial_{v_1} z_q & \dots\dots & \partial_{v_1}\partial_v^{\boldsymbol{\alpha}} z_1 & \dots & \partial_{v_1}\partial_v^{\boldsymbol{\alpha}} z_q \\
0 & 1 & \dots & 0 & \partial_{v_2} z_1 & \dots & \partial_{v_2} z_q & \dots\dots & \partial_{v_2}\partial_v^{\boldsymbol{\alpha}} z_1 & \dots & \partial_{v_1}\partial_v^{\boldsymbol{\alpha}} z_q \\
\dots & \dots & & \dots & \dots & & \dots & & \dots & & \dots \\
0 & 0 & \dots & 1 & \partial_{v_n} z_1 & \dots & \partial_{v_n} z_q & \dots\dots & \partial_{v_n}\partial_v^{\boldsymbol{\alpha}} z_1 & \dots & \partial_{v_n}\partial_v^{\boldsymbol{\alpha}} z_q \\
\nu_1^1 & \nu_2^1 & \dots & \nu_q^1 & \sigma_{0,1}^1 & \dots & \sigma_{0,q}^1 & \dots\dots & \sigma_{\boldsymbol{\alpha},1}^1 & \dots & \sigma_{\boldsymbol{\alpha},q}^1 \\
\dots & \dots & & \dots & \dots & & \dots & & \dots & & \dots \\
\nu_1^k & \nu_2^k & \dots & \nu_q^k & \sigma_{0,1}^k & \dots & \sigma_{0,q}^k & \dots\dots & \sigma_{\boldsymbol{\alpha},1}^k & \dots & \sigma_{\boldsymbol{\alpha},q}^k
\end{pmatrix}.$$

Here instead of letting $z \in P^r(\mathbb{R}^n, \mathbb{R}^q)$, we allow $z \in P^s(\mathbb{R}^n, \mathbb{R}^q)$ where $s > r$. When $s = r + 1$ the coefficients $\partial_{v_i}\partial_v^{\boldsymbol{\alpha}} z$ are linearly independent. Hence by Sard's theorem the sections $z \in P^s(\mathbb{R}^n, \mathbb{R}^q)$ such that $z(D) \pitchfork \Sigma$ is dense. This proves the theorem. □

Definition 1.3. *A closed subset A in a manifold V is stratified if $A = \bigcup_{j\le N} A_j$ where A_j's are locally closed submanifolds, and*

$$\bar{A}_k = \bigcup_{j=k}^{N} A_j, \ \forall\, 0 \le k \le N.$$

If A is stratified by a union of simplices of some triangulation of the manifold, A is called a polyhedron.

It is easy to generalize the Thom transversality theorem to the case of stratified subspaces, where now transversality means transversality to each strata.

2. Holonomic Approximation

2.1. **Statement and Proof.** A natural question for jet bundles is whether any section can be approximated by a holonomic section. In general this is not true. For example, take the standard inclusion $\mathrm{Op}\, S^1 \subset \mathbb{R}^2$ together with the line field $r\partial_r$. This defines a section in $J^1(\mathrm{Op}\, S^1, \mathbb{R}^2)$ that cannot be approximated by holonomic sections. In addition, even after a C^1-small diffeotopy, the section can still not be approximated. However, we have the following theorem:

Theorem 2.1 (Holonomic Approximation Theorem). *Let $A \subset V$ be a polyhedron of positive codimension and $F : \mathrm{Op}\, A \to X^{(r)}$ a section. Then for any $\delta, \epsilon > 0$ there is a diffeotopy*

$$h^\tau : V \to V,\ \tau \in [0,1];\quad \|h^\tau - \mathrm{id}\| \le \delta,\ \forall\, \tau \in [0,1],$$

and a holonomic section $\tilde{F} : \mathrm{Op}\, h^1(A) \to X^{(r)}$ such that

$$\mathrm{dist}(\tilde{F}(v), F|_{\mathrm{Op}\, h^1(A)}(v)) < \epsilon,\ \forall\, v \in \mathrm{Op}\, h^1(A).$$

Here $\mathrm{Op}\, A$ means some open neighbourhood of A. One should be aware that the $\mathrm{Op}\, h^1(A)$ where $\tilde{F}$ is defined is contained in $\mathrm{Op}\, A$ (meaning it is usually not of the same size as $h^1(\mathrm{Op}\, A)$, where $\mathrm{Op}\, A$ is the subset where F is defined).

Theorem 2.2 (Parametric Holonomic Approximation). *Let $A \subset V$ be a polyhedron of positive codimension and $F_z : \mathrm{Op}\, A \to X^{(r)}$ a family of sections parametrized by $z \in I^m$ such that F_z is holonomic for all $z \in \partial I^m$. Then for any $\delta, \epsilon > 0$ there is a family of diffeotopies*

$$h_z^\tau : V \to V,\ \tau \in [0,1];\quad \|h_z^\tau - \mathrm{id}\| \le \delta,\ \forall\, \tau \in [0,1],\ z \in I^m,$$

and a family of holonomic sections $\tilde{F}_z : \mathrm{Op}\, h_z^1(A) \to X^{(r)}$ such that

$$h_z^\tau = \mathrm{id},\ \tilde{F}_z = F_z,\ \forall\, z \in \mathrm{Op}\, \partial I^m;$$

$$\mathrm{dist}(\tilde{F}_z(v), F_z|_{\mathrm{Op}\, h_z^1(A)}(v)) < \epsilon,\ \forall\, v \in \mathrm{Op}\, h_z^1(A),\ z \in I^m.$$

Consider a triangulation of the manifold V such that A is a subcomplex. Then since the fibration over a simplex must be trivial, by induction on the skeleton of A, the theorem can be deduced from the following proposition:

Theorem 2.3. *Let $I^k \subset \mathbb{R}^n$ such that $k < n$ and $F : \mathrm{Op}\, I^k \to J^r(\mathbb{R}^n, \mathbb{R}^q)$ a section that is holonomic over $\mathrm{Op}\, \partial I^k$. Then for any $\delta, \epsilon > 0$ there is a diffeomorphism*

$$h : \mathbb{R}^n \to \mathbb{R}^n,\ h(x_1, ..., x_n) = (x_1, ..., x_{n-1}, x_n + \varphi(x_1, ..., x_n)),$$

and a holonomic section $\tilde{F} : \mathrm{Op}\, h(I^k) \to J^r(\mathbb{R}^n, \mathbb{R}^q)$ such that

$$h|_{\mathrm{Op}\, \partial I^k} = \mathrm{id},\ \tilde{F}|_{\mathrm{Op}\, \partial I^k} = F|_{\mathrm{Op}\, \partial I^k};$$

$$\mathrm{dist}(\tilde{F}(v), F|_{\mathrm{Op}\, h(I^k)}(v)) < \epsilon.$$

The theorem will be proved by induction. For simplicity, we introduce the following definition to describe the behaviour when by induction hypothesis the section is adjusted to be holonomic on low dimensional slices.

Definition 2.1. *A section $F : V \to X^{(r)}$ is holonomic over $A \subset V$ if $F|_A$ is holonomic: there is $\tilde{F} : \mathrm{Op}\, A \to X^{(r)}, \tilde{F}|_A = F|_A$.*

Given a fibration $\pi : V \to B$, $F : V \to X^{(r)}$ is fiberwise holonomic if there is a continuous family of holonomic extensions

$$\tilde{F}_b : \mathrm{Op}\, \pi^{-1}(b) \to X^{(r)},\ \tilde{F}_b|_{\pi^{-1}(b)} = F|_{\pi^{-1}(b)},\ b \in B.$$

Here $\tilde{F}_b$ a continuous family means that there is continuous family

$$\tilde{F} : \mathrm{Op}\, \mathrm{Gr}(\pi) \to X^{(r)} \times B,\ \tilde{F}|_{\mathrm{Op}\, \mathrm{Gr}(\pi) \cap V \times \{b\}} = \tilde{F}_b.$$

Where $\mathrm{Gr}(\pi) = \{(v, \pi(v)) | v \in V\} \subset V \times B$.

Note that the space of holonomic extensions is convex, and hence contractible.

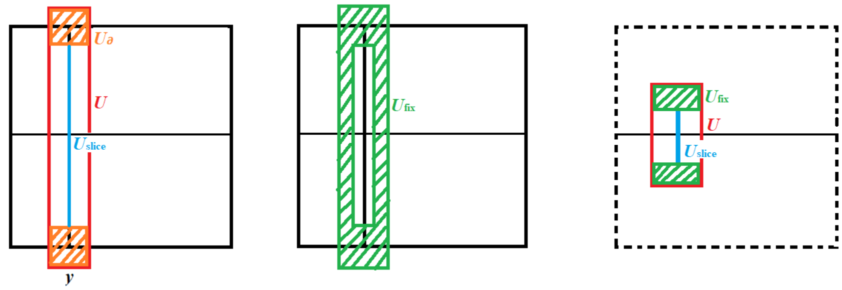

FIGURE 1. $U(y), U^\partial(y), U^{\text{slice}}(y)$ and $U^{\text{fix}}(y)$; The left two figures are pictures in $I^l \times I^{k-l} \times 0$ when $l = k - l = 1$, and the right one is the picture in $I^l \times 0 \times I^{n-k}$ when $l = n - k = 1$.

Lemma 2.4. *For closed subsets $B \subset A \subset V$ and a section $F : \operatorname{Op} A \to X^{(r)}$ being holonomic over $\operatorname{Op} B$, there exists a family of holonomic sections*

$$\tilde{F}_v : \operatorname{Op}\{v\} \to X^{(r)},\ \tilde{F}_v(v) = F(v),\ v \in A;$$
$$\tilde{F}_v = F|_{\operatorname{Op}\{v\}},\ v \in B.$$

Proof. For $v \in B$, set $\tilde{F}_v = F$. For $v \in A \backslash B$, take the Taylor polynomial map corresponding to $F(v)$, then this defines a holonomic extension $\tilde{F}_v$. When $v \in A \backslash B$, the family is continuous since

$$\tilde{F}_v(w) = \tilde{F}(v, w) = \sum_{|\boldsymbol{\alpha}| \le r} \frac{1}{\boldsymbol{\alpha}!} F_{\boldsymbol{\alpha}}(v)(w - v)^{\boldsymbol{\alpha}},\ v, w \in A \backslash B.$$

Consider a bump function $\chi : A \to [0, 1]$ such that $\chi|_B = 1, \chi|_{A \backslash \operatorname{Op} B} = 0$. Now to make the family of holonomic extensions continuous near B, we define

$$\tilde{F}_v(w) = \chi(v)F(w) + (1 - \chi(v)) \sum_{|\boldsymbol{\alpha}| \le r} \frac{1}{\boldsymbol{\alpha}!} F_{\boldsymbol{\alpha}}(v)(w - v)^{\boldsymbol{\alpha}}.$$

This turns out to be a continuous family of holonomic sections that satisfy the requirements. $\square$

Now we prove the holonomic approximation theorem for cubes by induction. Namely the induction lemma is as follows:

Lemma 2.5. *Let $I^k \subset \mathbb{R}^n$. Suppose a section $F : \operatorname{Op} I^k \to J^r(\mathbb{R}^n, \mathbb{R}^q)$ is holonomic over $\operatorname{Op} \partial I^k$ and is fiberwise holonomic with respect to the fibration $\pi_{k-l} : I^k \to I^{k-l}$. There is a diffeomorphism*

$$h : \mathbb{R}^n \to \mathbb{R}^n;\ h(x_1, ..., x_n) = (x_1, ..., x_{n-1}, x_n + \varphi(x_1, ..., x_n))$$

that $\|\varphi\|_{C^0} < \delta$, and a section $\tilde{F} : \operatorname{Op} h(I^k) \to J^r(\mathbb{R}^n, \mathbb{R}^q)$ such that
(1). $h = \mathrm{id}$, $\tilde{F} = F$ on $\operatorname{Op} \partial I^k$;
(2). $\|\tilde{F} - F|_{\operatorname{Op} h(I^k)}\|_{C^0} < \epsilon$;
(3). $\tilde{F}|_{h(I^k)}$ is fiberwise holonomic with respect to the fibration $\pi_{k-l-1} : h(I^k) \to I^{k-l-1}$.

Before starting the proof, we fix some notations. Given $A \subset \mathbb{R}^n$, $N_\delta(A)$ is the cubical δ-neighbourhood (i.e. δ-neighbourhood under the norm $\|x\|_\infty = \max_{1 \le i \le n} |x_i|$). Fix $\theta \in (0, 1)$ and $y \in I^{k-l} \subset I^k$. Now we set

$$U_\delta(y) = N_\delta(y \times I^l),\ U_\delta^\partial(y) = N_\delta(y \times \partial I^l),$$

$$U_\delta^{\text{slice}}(y) = (\bar{U}_{\theta\delta}(y) \backslash U_\delta^{\partial}(y)) \cap \pi_{k-l}^{-1}(y),\ U_\delta^{\text{fix}}(y) = (U_\delta(y) \backslash \bar{U}_{\theta\delta}(y)) \cup U_\delta^{\partial}(y).$$

For simplicity we might omit the subscription δ in the proof.

Roughly speaking, the idea is to consider enough pieces of the fiberwise extended holonomic sections F_i on $U_\delta(t_i)$, cut each of the extended section from the center $U_\delta^{\text{slice}}(t_i)$, and perturb F_i on the one of the two parts to meet the next piece holonomic section F_{i+1}. After this modification, we see that the different pieces of holonomic sections are glued together. Yet now the holonomic section is only defined away from where we cut, so we have to perturb I^k itself to the domain where $\tilde{F}$ is defined.

Proof. Without loss of generality we may assume that $l = k-1$, otherwise we can just set $t \in I$ be the first coordinate of $y \in I^{k-l}$ and copy the proof. Note that the continuity of the family F_t ensures that

$$\sup_{t\in I,\, x\in U(t+\sigma)\cap U(t)} |F_{t+\sigma}(x) - F_t(x)| \to 0,\ \sigma \to 0.$$

Therefore choose a bump function $\chi : I \to [0,1]$ such that $\chi|_{\operatorname{Op} U^{\text{slice}}(t)} = 1,\ \chi|_{U^{\text{fix}}(t)} = 0$, and $\|\chi\|_{C^r} \le C$. Now for $f_t = \operatorname{bs} F_t$ we define

$$f_t^\tau : U(t) \to \mathbb{R}^q,\ f_t^\tau(x) = \chi(t,x) f_{t+\tau}(x) + (1 - \chi(t,x)) f_t(x).$$

Write $F_t^\tau = J^r f_t^\tau$. Then we know by applying Leibniz rule for f_t^τ that for any $\epsilon > 0$, there exists $\sigma = 1/N$ sufficiently small such that

(1). $F_t^0 = F_t$, $t \in I$;
(2). $F_t^\tau|_{U^{\text{fix}}(t)} = F_t|_{U^{\text{fix}}(t)}$, $t \in I, \tau \in [0,\sigma]$;
(3). $F_t^\tau|_{\operatorname{Op}\{t\}\times I^{k-1}} = F_{t+\tau}|_{\operatorname{Op}\{t\}\times I^{k-1}}$, $t \in I, \tau \in [0,\sigma]$;
(4). $\|F_t^\tau - F_t\|_{C^0} < \epsilon$, $t \in I, \tau \in [0,\sigma]$.

Now for $0 \le i \le N$ set

$$A_i^{\text{glue}} = \{i\sigma\} \times I^{k-1}.$$

For $1 \le i \le N$ set

$$A_i^{\text{cut}} = U^{\text{slice}}(i\sigma - \sigma/2),$$
$$\Delta_i = ((i-1)\sigma, i\sigma),\ \Delta_i^- = ((i-1)\sigma, i\sigma - \sigma/2], \Delta_i^+ = [i\sigma - \sigma/2, i\sigma),$$
$$\tilde{U}_i = U(i\sigma) \cap \pi_1^{-1}(\Delta_i),\ \tilde{U}_i^- = U(i\sigma) \cap \pi_1^{-1}(\Delta_i^-),\ \tilde{U}_i^+ = U(i\sigma) \cap \pi_1^{-1}(\Delta_i^+),$$
$$\tilde{U}_i^{\text{cut}} = \tilde{U}_i^- \cap \tilde{U}_i^+ = U(i\sigma) \cap \pi_1^{-1}(i\sigma - \sigma/2).$$

Notice that $\tilde{U}_i^{\text{cut}} \backslash A_i^{\text{cut}} \subset U^{\text{fix}}(i\sigma)$. Hence the perturbation F_t^τ remains fixed on $\tilde{U}_i^{\text{cut}} \backslash A_i^{\text{cut}}$ for $\tau \in [0,\sigma]$, and $F_t^\sigma = F_{t+\sigma}$ on A_i^{glue}. Therefore

$$\tilde{F}(x) = \begin{cases} F_{i\sigma}(x), & x \in \tilde{U}_i^-, \\ F_{i\sigma}^\sigma(x), & x \in \tilde{U}_i^+, \end{cases} \quad 1 \le i \le N,$$

defines a holonomic section on $\Omega = \bigcup_{i=1}^N \tilde{U}_i \backslash A_i^{\text{cut}} \cup \bigcup_{i=0}^N \operatorname{Op} A_i^{\text{glue}}$. Now since there exists a diffeomorphism of the form

$$h : \mathbb{R}^n \to \mathbb{R}^n;\ h(x_1, ..., x_n) = (x_1, ..., x_{n-1}, x_n + \varphi(x_1, ..., x_n))$$

such that $h(I^k) \subset \Omega$, the lemma is proved. □

Proof of Theorem 2.3. As we said before, we prove the theorem by induction. When $l = 0$ this is lemma 2.4. Suppose we have constructed the section $\tilde{F}^l$ on $\operatorname{Op} h(I^k)$ being holonomic with respect to the fibration $\pi_{k-l} : h(I^k) \to I^{k-l}$. Now consider the section $\bar{F}^l = (h_*)^{-1}(\tilde{F}^l)$ defined over $\operatorname{Op} I^k$ that is still fiberwise holonomic. Now by lemma 2.5, there exists a section $\hat{F}^{l+1}$ that is fiberwise holonomic with respect to $\pi_{k-l-1} : h'(I^k) \to I^{k-l-1}$. Then $\tilde{F}^{l+1} = h_*(\hat{F}^{l+1})$ is the required section defined on $h(h'(I^k))$. This completes the proof. □

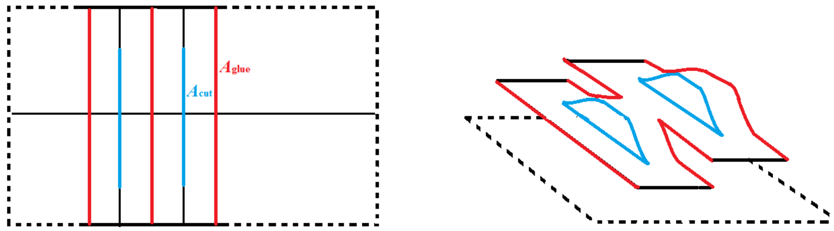

FIGURE 2. A_i^{cut} and A_i^{glue} in $I^l \times 0 \times I^{n-k}$ when $l = n - k = 1$.

2.2. Applications of Holonomic Approximation.

Definition 2.2. *Let E, F be vector bundles over V, $P \in \mathscr{D}^1(E,F)$ be a linear differential operator. The principal symbol of P is*

$$\sigma_P : T^*V \otimes E \to F,\ \ (\sigma_P(\xi))_x\,(s) = [P,f]_x(s),\ \ df_x = \xi_x.$$

Under local coordinates, if $P_\alpha^\beta = \sum_{1\le i\le n} a_{\alpha,i}^\beta(\partial/\partial x_i)$, then for a section $s : V \to E$,

$$\sigma_P(\xi)_\alpha^\beta = \sum_{1\le i\le n} a_{\alpha,i}^\beta \xi_i.$$

Consider the exterior differential $d : \wedge^{p-1}V \to \wedge^p V$. Note that $\left(\wedge^{p-1}V\right)^{(1)} \simeq \wedge^{p-1}V \otimes T^*V$. By tautological reasons one can easily know that it is in fact the composition

$$\wedge^{p-1}V \xrightarrow{J^1} \left(\wedge^{p-1}V\right)^{(1)} \xrightarrow{\sigma_d} \wedge^p V.$$

Therefore $\sigma_d : \left(\wedge^{p-1}V\right)^{(1)} \to \wedge^p V$ is actually a fibration where σ_d is just the antisymmetrization map. Now we know that any $\omega : V \to \wedge^p V$ there is a lift $F_\omega : V \to \left(\wedge^{p-1}M\right)^{(1)}$. We call F_ω a formal primitive of ω and we see that every p-form is formally exact. In addition for any $\alpha : V \to \wedge^{p-1}V$ there is a F_ω such that $\operatorname{bs} F_\omega = \alpha$.

Proposition 2.6 (Approximation by exact forms). *Let $A \subset V$ be a polyhedron with positive codimension and ω a p-form. Then there exists an arbitrarily C^0-small diffeotopy $h^\tau : V \to V$ such that ω can be approximated near $h^1(A)$ by an exact p-form $\tilde\omega = d\tilde\alpha$. Moreover, for any $(p-1)$-form α, one can choose $\tilde\alpha$ to be C^0-close to α near $h^1(A)$.*

Proposition 2.7 (Approximation by closed forms). *Let $A \subset V$ be a polyhedron with positive codimension and ω a p-form. Fix $a \in H^p(V;\mathbb{R})$. Then there exists an arbitrarily C^0-small diffeotopy $h^\tau : V \to V$ such that ω can be approximated near $h^1(A)$ by a closed p-form $\tilde\omega \in a$.*

Proof. Take any closed form $\Omega \in a$ and apply the previous proposition to the form $\omega - \Omega$. □

In addition, the parametric versions of these two propositions are also true. For the parametric case of approximation by closed forms, one may consider a family of closed forms $\{\Omega_u \in a\}_{u\in D^k}$ that extends $\{\omega_u \in a\}_{u\in S^{k-1}}$. Such extensions exist because the space of closed forms is convex.

Another application is on the approximation of tangential homotopies, which will be used in the approxiamtion of embeddings with certain tangential data. First we introduce some notations.

Definition 2.3. *Let $p : \mathrm{Gr}_n W \to W$ be the Grassmannian bundle whose fiber is the Grassmannian of n-planes in $T_w W$. Given an immersion $f : V \to W$, the tangential map is $\mathrm{G}\,df : V \to \mathrm{Gr}_n W;\ v \mapsto df(T_v V)$.*

A tangential homotopy is $F_t : V \to \mathrm{Gr}_n W$ where $F_0 = \mathrm{G}\,df$ and $p \circ F_t = f$ for any $t \in [0,1]$.

Theorem 2.8. *Let $A \subset V$ be a polyhedron of positive codimension and $F_t : V \to \mathrm{Gr}_n W$ a tangential homotopy. The given any $\delta, \epsilon > 0$ there is a diffeotopy $h^\tau : V \to V$ that is δ-close to identity in the C^0-topology, and an isotopy*

$$\tilde{f}_t : \mathrm{Op}\, h^1(A) \to W, \ \ \tilde{f}_0 = f|_{\mathrm{Op}\, h^1(A)}$$

such that the homotopy $\mathrm{G}\,d\tilde{f}_t$ is ϵ-close to the tangential homotopy $F_t|_{\mathrm{Op}\, h^1(A)}$.

Let X be the normal bundle of V, which is diffeomorphic to a tubular neighbourhood of V in W. Recall that $X^{(1)}$ can be viewed as the space of n-planes transverse to the fiber (non-vertical sections). Hence when the angles between $F_t(v)$ and $F_{t'}(v)$ are always less than $\pi/4$, the tangential maps F_t will be transverse to the fiber, which means they can be viewed as a homotopy of sections $F_t : V \to X^{(1)}$. In this case holonomic approximation applies.

Proof. First extend F_t to a tubular neighbourhood X. Divide the interval $I = \bigcup_{j=0}^{N-1} [j/N, (j+1)/N]$ so that on each subdivision $I_j = [j/N, (j+1)/N]$, the angles between $F_t(v)$ and $F_{t'}(v)$ are always less than $\pi/4$. By holonomic approximation on I_0 one can find a family of sections

$$f_{1,t} : \mathrm{Op}\, h_1^1(A) \to X, \ t \in I$$

so that $\mathrm{G}\,df_{1,t/N}$ is close to F_t on some $h_1^1(A)$, and denote $V_1 = f_{1,1}(\mathrm{Op}\, h_1^1(A))$, and $A_1 = h_1^1(A)$.

Suppose $f_{j-1,t}$ has been constructed. Then for I_j consider a tubular neighbourhood of V_{j-1} and the polyhedron $(A_{j-1}$. By holonomic approximation on can find

$$f_{j,t} : \mathrm{Op}\, h_j^1(A_{j-1}) \to X, \ t \in I$$

so that $\mathrm{G}\,df_{0,j/N+t/N}$ is close to F_t on some $h_j^1(A_{j-1})$. Now we can define the isotopy as follows

$$\tilde{f}_t = \begin{cases} f_{1,Nt}|_{\mathrm{Op}\, A}, & t \in I_0; \\ f_{2,Nt-1} \circ f_{1,1}|_{\mathrm{Op}\, A}, & t \in I_1; \\ \ldots & \ldots \\ f_{N,Nt-N+1} \circ \ldots \circ f_{1,2}|_{\mathrm{Op}\, A}, & t \in I_N. \end{cases}$$

Since there are only finite many steps, the approximation can be made arbitrarily close. □

3. h-Principle In General

3.1. **Partial Differential Relations.** In differntial topology, recall that an immersion is a map $f : M \to N$ such that df is injective at every point, and a submersion is a map $f : M \to N$ such that df is injective at every point. We may consider to translate these definitions into conditions on the 1-jet spaces. In general, it turns out that a lot of conditions for smooth mappings in topology and geometry can be described as conditions of sections on the jet bundles.

Definition 3.1. *A partial differential relation $\mathcal{R}$ is a subset in $X^{(r)}$. The relation is closed (open) if $\mathcal{R}$ is a closed (open) subset. It is determined (overdetermined / underdetermined) if $\mathrm{codim}\mathcal{R} = \dim V$ ($\mathrm{codim}\mathcal{R} < \dim V$ / $\mathrm{codim}\mathcal{R} > \dim V$).*

The immersion relation $\mathcal{R}_{\text{imm}} \subset J^1(V, W)$ over each point $(v, w) \in V \times W$ consists of all injections in $\text{Hom}(T_v V, T_w W)$, and the submersion relation $\mathcal{R}_{\text{sub}} \subset J^1(V, W)$ over each point $(v, w) \in V \times W$ consists of all injections in $\text{Hom}(T_v V, T_w W)$. They are both open and underdetermined.

The closed relation $\mathcal{R}_{\text{cl}}^k \subset \left(\wedge^k V\right)^{(1)}$ at each point $(v, \alpha) \in \wedge^k V$ consists of all tensors $\beta \in \wedge^k V \otimes \wedge^1 V$ that is symmetric (whose image under the natural projection $\wedge^k V \otimes \wedge^1 V \to \wedge^{k+1} V$ vanishes). It is closed and underdetermined.

Definition 3.2. *A formal solution for the differential relation $\mathcal{R} \subset X^{(r)}$ is a section $F : V \to \mathcal{R} \subset X^{(r)}$. A genuine solution for $\mathcal{R}$ is a section $f : V \to X$ such that $J^r f : V \to \mathcal{R} \subset X^{(r)}$, and $J^r f$ is called an r-solution.*

For example, the genuine solutions for $\mathcal{R}_{\text{imm}}$ ($\mathcal{R}_{\text{sub}}$) are immersions (submersions) $f : V \to W$, the genuine solutions for $\mathcal{R}_{\text{cl}}^k$ are closed k-forms, and the genuine solution for $\mathcal{R}_{\text{iso}}$ are isometries $f : (V, g_V) \to (W, g_W)$.

3.2. **Homotopy Principles.** We know that existence of formal solutions is a necessary condition for the existence of genuine solutions. Although in general it cannot automatically imply the existence of genuine solutions, it turns out that in a number of important cases, the existence of formal solutions will indeed imply the existence of genuine solutions.

Definition 3.3. *A differential relation $\mathcal{R}$ satisfies the h-principle if every formal solution is homotopic in* $\text{Sec}\mathcal{R}$ *to a genuine solution.*

The differential relation $\mathcal{R}$ satisfies the parametric h-principle if for every family of sections

$$\varphi_0 : (D^k, S^{k-1}) \to (\text{Sec}\mathcal{R}, \text{Hol}\mathcal{R}),\ k \in \mathbb{N},$$

there exists a homotopy with fixed boundary

$$\varphi_t : (D^k, S^{k-1}) \to (\text{Sec}\mathcal{R}, \text{Hol}\mathcal{R}),\ t \in [0, 1],\ k \in \mathbb{N},$$

such that $\varphi_1(D^k) \subset \text{Hol}\mathcal{R}$. In other words, $\text{Hol}\mathcal{R} \subset \text{Sec}\mathcal{R}$ *is a weak homotopy equivalence.*

The differential relation $\mathcal{R}$ satisfies the local h-principle near $A \subset V$ if every formal solution $F : \text{Op}\, A \to \mathcal{R}$ is homotopic to a holonomic section on $\text{Op}\, A$.

The differential relation $\mathcal{R}$ satisfies the reative h-principle near (A, B) for $B \subset A \subset V$ if every formal solution $F : \text{Op}\, A \to \mathcal{R}$ that is holonomic on $\text{Op}\, B$ is homotopic relative B to a genuine solution.

The differential relation $\mathcal{R}$ satisfies the C^0-dense h-principle if for every formal solution F and any neighbourhood $U \subset X$ of $\text{bs}F(V)$*, F is homotopic through sections in U to a genuine solution.*

The following theorem is a 1-parametric version of h-principle for immersions, which gives an example where h-principle holds, showing that h-principle actually holds in some important cases we care about.

Proposition 3.1. *Let $f_0, f_1 : S^1 \to \mathbb{R}^2$ be two immersions with the same winding number*

$$\deg(df_0/|df_0|) = \deg(df_1/|df_1|).$$

Then they are homotopic through a family of immersions $f_t : S^1 \to \mathbb{R}^2$.

Consider the immersion relation $\mathcal{R}_{\text{imm}} \subset J^1(S^1, \mathbb{R}^2) = (S^1 \times \mathbb{R}^2) \oplus (S^1 \times \mathbb{R}^2)$ is the subset $(S^1 \times \mathbb{R}^2) \oplus (S^1 \times \mathbb{R}^2 \backslash \{0\})$, being homotopic equivalent to $S^1 \times S^1$. The sections $F : S^1 \to \mathcal{R}_{\text{imm}}$ is uniquely determined by the mapping degree of F composed with the projection $p_2 : (S^1 \times \mathbb{R}^2) \oplus (S^1 \times \mathbb{R}^2 \backslash \{0\}) \to S^1 \times S^1 \to S^1$. Hence any two formal solutions are homotopic if and only if they have the same winding number.

Proof. First consider some tubular neighbourhood of S^1 in $\mathbb{R}^2$, extend f_0, f_1 to sections $\operatorname{Op} S^1 \to \mathbb{R}^2$, and extend the formal homotopy F_t to a family of sections $\operatorname{Op} S^1 \to J^1(\operatorname{Op} S^1, \mathbb{R}^2)$. Then apply the parametric holonomic approximation theorem to get a genuine homotopy that is equal to f_0, f_1 on the boundary. $\square$

3.3. h**-Principle for Immersions/Submersions.** Recall that a manifold V is open if there are no closed manifolds being its components. Here note that we are also allowing manifolds with boundary. We first show some basic properties for open manifolds.

Definition 3.4. *Let V be an open manifold. A proper path $\gamma : [0, \infty) \to V$ connects $\gamma(0)$ with ∞ if $\gamma(\infty) \in \partial V$ or $\gamma(\infty)$ does not exist.*

Definition 3.5. *For a manifold V, $K \subset V$ is a core if K is a polyhedron of positive codimension and for any neighbourhood U of K, there exists an isotopy $\varphi_t : V \to V$ such that $\varphi_t|_K = \mathrm{id}_K$ and $\varphi_1(V) \to U$.*

Proposition 3.2. *For an open manifold V, there exists a core $K \subset V$.*

Proof. Fix a triangulation of V. For any barycenter v_α of some n-simplex, connect v_α with ∞ by a path γ_α such that $\{\gamma_\alpha\}_{\alpha \in I}$ are pairwise disjoint, for any $(n-1)$-simplex σ, γ_α intersects σ at most once, and for any simplex σ of $\dim \sigma \leq n-2$, γ_α and σ are disjoint. Namely first by simplicial approximation, we can assume γ_α lie in the 1-skeleton of the barycenter subdivision. Then consider a tubular neighbourhood of the 1-skeleton. Now by Sard theorem it is easy to perturb the curves γ_α along some normal direction so that they satisfy these conditions.

Apply tubular neighbourhood theorem we will get a tubular neighbourhood U_Γ for $\Gamma = \bigcup_{\alpha \in I} \gamma_\alpha([0,1])$. Suppose Then we know that the complement of U_Γ deformation retracts to the $(n-1)$-polyhedron

$$K = \mathrm{sk}_{n-1} V \backslash \bigcup_{\sigma^{n-1} \cup U_\Gamma \neq \emptyset} \sigma.$$

via an isotopy. One the other hand, V deformation retracts to $V \backslash U_\Gamma$ via an isotopy. Hence K is the core of V. $\square$

Another proof is to use Morse theory. Fix a proper Morse function f on V such that all critical points are in different level sets. Now suppose $p, q \in \operatorname{Crit}(f)$, $f(p) < f(q)$. If $\operatorname{ind}(p) > \operatorname{ind}(q)$, then on some level set between $f(p)$ and $f(q)$, by Sard theorem the stable sphere S_p^s and the unstable sphere S_q^u will generically be disjoint. Hence one can consider two disjoint neighbourhoods U_p, U_q of trajectories passing through p and q, and adjust the gradient vector field independently to make $f(p) > f(q)$. In addition we may assume that $f(p) = \operatorname{ind}(p) + 1/2$. Now the Morse function f is what we call self-indexing. For the complex

$$CM^k(V) = H_k(f^{-1}([k, k+1]), f^{-1}(k)),$$

$$\partial_M : H_k(f^{-1}([k, k+1]), f^{-1}(k)) \xrightarrow{\delta_*} H_{k-1}(f^{-1}(k)) \xrightarrow{i_*} H_{k-1}(f^{-1}([k-1, k]), f^{-1}(k-1)),$$

we can show that $H^*(V; \mathbb{Z}) \simeq HM^*(V; \mathbb{Z})$. In addition one can show that

$$CM^k(V) = \bigoplus_{\operatorname{ind}(p)=k} \mathbb{Z}p, \ \ \partial_M p = \sum_{\operatorname{ind}(q)=k-1} [S_p^u] \cdot [S_q^s]\, q.$$

Hence as $H_n(V) = 0$, every index-n critical point is joined with an index-$(n-1)$ critical point. Now apply the cancelation theorem for critical points of Morse functions there will be no index-n critical points. Therefore let

$$K = \bigcap_{t>0} \varphi_t(V) = \bigcup_{p \in \operatorname{Crit}(f)} D_p^s.$$

This is also the core for V. For the details one can check Milnor, *Lectures on the h-Cobordism Theorem*, Section 7.

Now we will consider a certain case when h-principle works on open manifolds called the Diff V-invariant relation. This includes immersion, submersion and directed immersion relations. Let $\mathrm{Diff}_V X$ be the diffeomorphisms f_X of X so that there is some $f_V \in \mathrm{Diff}\, V$, $p \circ f_X = f_V \circ p$. This map $f_X \mapsto f_V$ defines a fibration $\mathrm{Diff}_V X \to \mathrm{Diff}\, V$.

Definition 3.6. *For a fibration $p : X \to V$, fix some section $j : \mathrm{Diff}\, V \to \mathrm{Diff}_V X$. A differential relation $\mathcal{R} \subset X^{(r)}$ is* Diff V*-invariant if the action*

$$h_* : (v, J^r s(v)) \mapsto (h(v), J^r(j(h) \circ s)(v))$$

leaves $\mathcal{R}$ invariant.

In the case when $X = J^r(V, W)$, a canonical section $j : \mathrm{Diff}\, V \to \mathrm{Diff}_V J^r(V, W)$ can be defined by

$$j(h) : (v, s(v)) \mapsto (h(v), h_* s(v))$$

and h_* is just the usual pushforward map. This is the only case we care about in the following part of the section.

It is easy to check that immersions, submersions and directed immersions are all Diff V-invariant relations. Here for $A \subset \mathrm{Gr}_n W$ the differential relation $\mathcal{R}_A$ for A-directed embedding is defined at each fiber over $(v, w) \in V \times W$ by the maps $f \in \mathrm{Hom}(T_v V, T_w W)$ such that $f(T_v V) \subset A$.

Lemma 3.3 (Local h-principle for Diff V-invariant relations)**.** *Let $X \to V$ be a fibration. Then any* Diff V*-invariant relation $\mathcal{R}$ satisfies all finds of local h-principles for all positive codimensional polyhedrons $A \subset V$.*

Proof. We show the nonparametric case, that is, $\pi_0(\mathrm{Sec}_{\mathrm{Op}\, A}\mathcal{R}, \mathrm{Hol}_{\mathrm{Op}\, A}\mathcal{R}) = 0$. The proof for the general case will be the same.

Given $F \in \mathrm{Sec}_{\mathrm{Op}\, A}\mathcal{R}$, by holonomic approximation theorem, there exists a C^0-small diffeotopy $h^\tau : V \to V$ and $\hat{F} \in \mathrm{Hol}_{\mathrm{Op}\, h^1(A)}\mathcal{R}$ that is C^0-close to F. Now let $\tilde{F} = (h^1)^* \hat{F}$. Then $\tilde{F} \in \mathrm{Hol}_{\mathrm{Op}\, A}\mathcal{R}$ as $\mathcal{R}$ is Diff V-invariant. Since h^1 is C^0-small and $\hat{F}$ is C^0-close to F, we can conclude that $\tilde{F}$ is C^0-close to F. This also implies the C^0-dense h-principle. $\square$

Theorem 3.4 (h-Principle for Diff V-invariant relations on open manifolds)**.** *Let $X \to V$ be a fibration and V be an open manifold. Then any* Diff V*-invariant relation $\mathcal{R}$ satisfies all finds of h-principles.*

Proof. We only show the nonparametric h-principle. Let $K \subset V$ be the core of V. Then there exists $\varphi_t : V \to V$ such that $\varphi_t|_K = \mathrm{id}|_K$ and $\varphi^1(V) \subset \mathrm{Op}\, K$. The local h-principle for $\mathcal{R}$ shows that any $(\varphi^1)_* F \in \mathrm{Sec}_{\mathrm{Op}\, K}\mathcal{R}$ is homotopic to $\hat{F} \in \mathrm{Hol}_{\mathrm{Op}\, K}\mathcal{R}$. Now Since $\mathcal{R}$ is Diff V-invariant, we can conclude that $(\varphi^1)^* \hat{F}$ is the holonomic section that is required. $\square$

Corollary 3.5 (Gromov, 1986)**.** *Let $\mathcal{R} \subset J^1(V, W)$ be the immersion/submersion/directed immersion relation where V is an open manifold. Then $\mathcal{R}$ satisfies all types of h-principles.*

While for general closed manifolds and the immersion/submersion relation, h-principle does not hold, we can reformulate problems about closed manifolds in terms of open manifolds by the microextension trick, which is developed by M. Hirsch.

Theorem 3.6 (Hirsch, 1959)**.** *The parametric C^0-dense h-principle holds for immersion relation $\mathcal{R}_{imm} \subset J^1(V, W)$ if $\dim V < \dim W$.*

Proof. In the nonparametric case let F be a formal solution for $\mathcal{R}_{\text{imm}}$ and $f = \operatorname{bs} F$. Then F determines a monomorphism $TV \to f^*TW$. Fix a Riemannian metric and let νV be the normal bundle of TV in f^*TW and $\pi : \nu V \to V$ be the projection. Then $T(\nu V) = \pi^*TV \oplus \pi^*\nu V$, so F lifts to

$$\tilde{F} : T(\nu V) \xrightarrow{\sim} \pi^*TV \oplus \pi^*\nu V \to TV \oplus \nu V \xrightarrow{\sim} f^*TW.$$

Then the C^0-dense h-principle for immersions $V \to W$ comes from the one for immersions $\nu V \to W$.

In the parametric case where there is a family of formal solutions $F_z\,(z \in D^k)$, consider the normal bundle $\nu_0 V$ with respect to F_0. As before we can define $\tilde{F}_0 : T(\nu_0 N) \to TW$. Since D^k is contractible, we can easily extend $\tilde{F}_0$ to a family $\tilde{F}_z$. Then the conclusion follows. $\square$

3.4. ***h*-Principle for Directed Embeddings.** As we have considered directed immersions in the previous section, in this section we consider directed embeddings. First we discuss the h-principle for open manifolds. The following theorem follows from the approximation of tangential homotopies.

Theorem 3.7 (h-principle for A-directed embeddings on open manifolds)**.** *Let $A \subset \operatorname{Gr}_n W$ be an open subset and $f_0 : V \to W$ an embedding with $df_0 : TV \to TW$ being homotopic to $F_1 : TV \to TW$ where $\operatorname{G}F_1 : V \to A \subset \operatorname{Gr}_n W$. Then f_0 can be isotoped to an A-directed embedding $f_1 : V \to W$, and F_1 will be homotopic to df_1 through F_t such that $\operatorname{bs} F_t = f_t$ and $\operatorname{G}F_t : V \to A \subset \operatorname{Gr}_n W$.*

Proof. Let $K \subset V$ be a core of V. Then by approximation of tangential homotopy we are able to approximate $\operatorname{G}F_t$ by a family $\tilde{f}_t : \operatorname{Op} h^1(K) \to W$. Since $A \subset \operatorname{Gr}_n W$ is open we can choose the approximation such that $\operatorname{G} d\tilde{f}_t$ is sufficiently close to $\operatorname{G}F_t$. Now compress V via φ_t to $\operatorname{Op} h^1(K)$ and then compose with $\tilde{f}_t$. This gives the required isotopy.

It suffices to construct such a homotopy between $F_1|_{\operatorname{Op} h^1(K)}$ and $d\tilde{f}_1$ such that $\operatorname{bs} F_t = f_t$ and $\operatorname{G}F_t : V \to A \subset \operatorname{Gr}_n W$. Since $\operatorname{G}F_1|_{\operatorname{Op} h^1(K)}$ is C^0-close to $\operatorname{G} d\tilde{f}_1$, there exists a tangential homotopy G_t between them such that $\pi \circ G_t = f_t$. In general, a tangential map to $\operatorname{Gr}_n W$ does not canonically lift to a map to Stiefel bundle $\operatorname{St}_n W$ (whose fiber at each point is the monomorphisms $F_v : T_v V \to T_{f(v)} W$). However, since $\operatorname{G}F_t|_{\operatorname{Op} h^1(K)}$ is C^0-close to $\operatorname{G} d\tilde{f}_t$, the tangential homotopy can be lifted to a homotopy $F_t : TV \to TW$. $\square$

In general, the h-principle for directed embeddings of closed manifolds does not hold. Hence we need some additional assumptions. Roughly speaking, it is necessary to enable the embedding to shift in extra dimensions.

Definition 3.7. *Let $n < m \le q$. An open subset $A \subset \operatorname{Gr}_n W$ is m-complete if there exists an open set $\hat{A} \subset \operatorname{Gr}_m W$ such that $A = \bigcup_{\hat{L} \in \hat{A}} \operatorname{Gr}_n \hat{L}$.*

Let $A \subset \operatorname{Gr}_n \mathbb{R}^q$ be the space of n-planes that intersects $\{v\} \times \mathbb{R}^{q-m}$ transversely. Then A is m-complete if we consider $\hat{A}$ consisting of all m-planes such that intersects $\{v\} \times \mathbb{R}^{q-m}$ trivially.

Theorem 3.8 (h-principle for A-directed embeddings of closed manifolds)**.** *Let $A \subset \operatorname{Gr}_n W$ be an open subset being m-complete for some $n < m \le q$. Suppose $f_0 : V \to W$ is an embedding with $df_0 : TV \to TW$ being homotopic to $F_1 : TV \to TW$ where $\operatorname{G}F_1 : V \to A \subset \operatorname{Gr}_n W$. Then f_0 can be isotoped to an A-directed embedding $f_1 : V \to W$, and F_1 will be homotopic to df_1 through F_t such that $\operatorname{bs} F_t = f_t$ and $\operatorname{G}F_t : V \to A \subset \operatorname{Gr}_n W$.*

Proof. Denote by $\mathrm{Gr}_{m,n}W$ the manifold of all (m,n)-flags in W, that i, pairs of tangent planes (L^m, L^n) such that $L^n \subset L^m$. Let $\hat{\pi} : \mathrm{Gr}_{m,n}W \to \mathrm{Gr}_m W$ and $\pi : \mathrm{Gr}_{m,n}W \to \mathrm{Gr}_n W$ be the natural projections, and $\bar{A} = \{(\hat{L}, L) | \hat{L} \subset \hat{A}, L \subset \mathrm{Gr}_n \hat{L}\}$. Then $\hat{\pi}(\bar{A}) = \hat{A}, \pi(\bar{A}) = A$.

Consider a triangulation of V such that on every n-simplex the map $\mathrm{G}F_1 : I^n \to \mathrm{Gr}_n W$ lifts to $\bar{G}_1 : I^n \to \bar{A} \subset \mathrm{Gr}_{m,n}W$. Set $K = \mathrm{sk}_{n-1}V$. Then by the theorem in the open case we can construct a required homotopy on $\mathrm{Op}\, K$. Now it suffices to construct a homotopy f_t on I^n being fixed on $\mathrm{Op}\, \partial I^n$.

Now since $\mathrm{G}F_1 : I^n \to \mathrm{Gr}_n W$ lifts to $\bar{G}_1 : I^n \to \bar{A} \subset \mathrm{Gr}_{m,n}W$, we know that $\mathrm{G}F_t$ also lifts to $\bar{G}_t$. Now set $\hat{G}_t = \hat{\pi} \circ \bar{G}_t$. Now we consider the microextension trick. Let νI^n be the normal bundle of I^n such that $\hat{G}_t(v) = G_t(v) \oplus \nu_v I^n$. Then f_0 can be extended to $\hat{f}_0 : \nu I^n \to W$ such that $\mathrm{G}\, d\hat{f}_0|_V = \hat{G}_0$. By the h-principle in the open case we can construct an isotopy $\hat{f}_t : \nu I^n \to W$ such that $\hat{f}_1$ is $\hat{A}$-directed embedding. Then the restriction $f_t = \hat{f}_t|_V$ is an isotopy from f_0 to an A-directed embedding $f_1 : I^n \to W$. □

3.5. **Integrability/Microflexibility.** Up till now we have only talked about open differential relations, where it is (relatively) easy to find enough formal solutions. In practice there are a lot of interesting differential relations that are not open. However in order to get some type of h-principles, one still needs some condition to ensure that there are enough formal solutions.

Definition 3.8. *A differential relation $\mathcal{R} \subset X^{(r)}$ is locally integrable if for any $v \in V, F \in \mathcal{R}$, there is a holonomic extension $\tilde{F} : \mathrm{Op}\,\{v\} \to \mathcal{R}$.*

The local integrability condition is imposed to ensure that we can do holonomic extension in $\mathcal{R}$, which is the key to Proposition 2.4. Of course open relations are locally integrable. Besides open relations, the isotropic immersion relations into symplectic/contact manifolds are also locally integrable, and we will show this in Section 8.

Definition 3.9. *Let $\theta_k = (I^n, I^k \cup \partial I^n)$. A pair $(A, B) \subset V$ is called a θ_k-pair if it is diffeomorphic to θ_k.*

Definition 3.10. *A differential relation $\mathcal{R} \subset X^{(r)}$ is k-microflexible if for any small enough neighbourhood $U \subset V$, given*

(1). θ_k-pair $(A, B) \subset U$;

(2). holonomic section $F^0 : \mathrm{Op}\, A \to \mathcal{R}$;

(3). holonomic homotopy $F^\tau : \mathrm{Op}\, B \to \mathcal{R}$, $\tau \in [0,1]$ over $\mathrm{Op}\, B$ that is constant over $\mathrm{Op}\,(\partial A \cap B)$;

there exists a number $\sigma > 0$ and a holonomic homotopy constant over $\mathrm{Op}\,(\partial A)$ that extends the homotopy $F^\tau : \mathrm{Op}\, B \to \mathcal{R}$ in $\tau \in [0, \sigma]$. $\mathcal{R} \subset X^{(r)}$ is microflexible if it is k-microflexible for any $0 \le k \le n$.

The microflexibility condition is imposed to allow perturbation and gluing of fiberwise holonomic sections as in Lemma 2.5. Namely, here A plays the role of $U(t)$, B plays the role of $U^{\mathrm{fix}}(t) \cup \mathrm{Op}\,(\{t\} \times I^{k-l})$, and $\partial A \cap B$ plays the role of $U^{\mathrm{fix}}(t)$.

Obviously, by Lemma 2.5, open relations are microflexible. Here we give one more example.

Lemma 3.9. *The differential relation $\mathcal{R}_{cl} \subset (\wedge^p V)^{(1)}$ defining closed p-forms is k-flexible for $k \neq p$.*

Proof. Showing microflexibility is just extending closed p-forms on $\mathrm{Op}\,(\partial I^n \cup I^k)$ to $\mathrm{Op}\, I^n$, and we know (from de Rham theory) that the obstruction is just $H^p(\partial I^n \cup I^k; \mathbb{R})$. □

Also we will show in Section 8 that isotropic/isosymplectic immersions into symplectic manifolds are k-microflexible if $k \neq 1$, and isotropic/isocontact immersions into contact manifolds are microflexible.

Now under the condition of integrability and microflexibility, there are no obstructions to extend the holonomic approximation theorem (Proposition 2.4 is ensured by integrability and Lemma 2.5 is ensured by microflexibility).

Theorem 3.10 (Holonomic $\mathcal{R}$-Approximation)**.** *Let $\mathcal{R} \subset X^{(r)}$ be a locally integrable and microflexible differential relation. Let $A \subset V$ be a polyhedron of positive codimension and $F : \operatorname{Op} A \to \mathcal{R}$ a section. Then for any $\delta, \epsilon > 0$ there is a diffeotopy*

$$h^\tau : V \to V,\ \tau \in [0,1];\quad \|h^\tau - \mathrm{id}\| \leq \delta,\ \forall\, \tau \in [0,1],$$

and a holonomic section $\tilde{F} : \operatorname{Op} h^1(A) \to \mathcal{R}$ such that

$$\operatorname{dist}(\tilde{F}(v), F|_{\operatorname{Op} h^1(A)}(v)) < \epsilon,\ \forall\, v \in \operatorname{Op} h^1(A).$$

Theorem 3.11 (Local h-principle)**.** *Let $X \to V$ be a fibration. Then any* $\mathrm{Diff}\, V$*-invariant locally integrable and microflexible relation $\mathcal{R}$ satisfies all finds of local h-principles for all positive codimensional polyhedrons $A \subset V$.*

Now we also want to extend the local h-principle since in many interesting cases, the differential relation may not be $\mathrm{Diff}\, V$-invariant, but only invariant under a much smaller subgroup.

Definition 3.11. *Let $G \subset \mathrm{Diff}_c V$ be a Lie subgroup, and $\mathfrak{g}$ its Lie algebra. G (or $\mathfrak{g}$) is capacious if*

(1). For any $\xi \in \mathfrak{g}$, any compact subset $K \subset V$ and any neighbourhood U of A, there is a vector field $\tilde{\xi}_{G,U} \in \mathfrak{g}$ that is supported in U and coincides with ξ on G;

(2). For any tangent hyperplane $\tau \in T_v V$, there exists a vector field $\xi \in \mathfrak{g}$ that is transverse to τ.

From the definition it is not hard to show that Hamiltonian diffeomorphism groups and contactomorphism groups are capacious. We will do this in Section 8.

Theorem 3.12 (Local h-principle)**.** *Let $X \to V$ be a fibration, and $G \subset \mathrm{Diff}_c V$ be a capacious subgroup. Then any G-invariant locally integrable and microflexible relation $\mathcal{R}$ satisfies all finds of local h-principles for all positive codimensional polyhedrons $A \subset V$.*

Proof. Recall the proof of the local h-principle, the only step that we need $\mathrm{Diff}\, V$-invariant condition is when we apply the diffeotopy h^τ in Theorem 3.10. However h^τ is not necessarily in G. This is the only problem here.

First notice that by condition (2) of capacious groups, since $\operatorname{codim}\mathcal{R} \geq 1$,one can always find a vector field $\xi \in \mathfrak{g}$ transverse to A. We work on a single simplex in A and choose a coordinate system so that $A = I^k$ so that $\xi = \partial/\partial x_n$.

Recall the proof of holonomic approximation, we cut the fiberwise holonomic sections along A_i^{cut}, perturb the section and glue along A_i^{glue}. Hence it suffices to construct a diffeotopy $h_i^\tau : \tilde{U}_i \to \tilde{U}_i$ so that

(1). $h_i^\tau \in G$;

(2). $h_i^1(I^k \cap \tilde{U}_i) \cap A_i^{\mathrm{cut}} = \emptyset$.

To construct the diffeotopy, the only thing we have in hand is $\xi = \partial/\partial x_n \in \mathfrak{g}$. Now we use property (1) of capacious groups, and find $\tilde{\xi}_i \in G$ that coincides with ξ on

$$U_{\delta/2}(A_i^{\mathrm{cut}}) \cap \{i/N + \sigma/4 \leq t \leq i/N + 3\sigma/4\}$$

and supported on $\tilde{U}_i$. Let h_i^τ be the flow of $(\delta/2)\tau\tilde{\xi}_i$. This completes the proof. □

4. Wrinkled Mappings/Embeddings

4.1. **h-Principle for Wrinkled Embeddings.** In the previous section when studying directed embeddings, we've seen that directed embeddings of closed manifolds does not have an h-principle in general. If we still force the formal embedding to move through a homotopy to a genuine one, singularities (that prevent us from getting a smooth embedding) will appear. The simplest singularities are of course cusps, and we study them in this section.

Definition 4.1. *A continuous map $f : V \to W$ is called a wrinkled embedding if*

(1). f is a topological embedding and smooth on an open dense subset;

(2). Singularities of f are exactly $(n-1)$-spheres S_j^{n-1} $(j \in I)$ that bound n-balls D_j^n;

(3). The map f near each sphere S_j^{n-1} is equivalent to the map $\operatorname{Op} S^{n-1} \to \mathbb{R}^m$

$$(y_1, ..., y_{n-1}, r) \mapsto \Big(y_1, ..., y_{n-1}, r^3 + 3(|y|^2 - 1)r, \int_0^r (r^2 + |y|^2 - 1)^2 dz, 0, ..., 0\Big).$$

A family of maps $f_t : V \to W$ is called a family of wrinkled embeddings if it is smooth away from embryos, where the nearby behaviour of f_t is equivalent to the family of map $\operatorname{Op}\{0\} \to \mathbb{R}^m$

$$(y_1, ..., y_{n-1}, r) \mapsto \Big(y_1, ..., y_{n-1}, r^3 + 3(|y|^2 - t)r, \int_0^r (r^2 + |y|^2 - t)^2 dz, 0, ..., 0\Big).$$

Theorem 4.1 (h-Principle for Wrinkled Embeddings; Eliashberg-Mishachev, 2011)**.** *Let $G_t : V \to \mathrm{Gr}_n W$ be a tangential rotation of an embedding $f_0 : V \to W$. Then there exists a homotopy of wrinkled embeddings $f_t : V \to W$ so that $\mathrm{G}\, df : V \to W$ is C^0-close to G_t. If the rotation G_t is constant on some subset, then the homotopy can also be chosen to be fixed on that subset.*

Just as what we did in approximation of tangential homotopies in Theorem, we may assume that the angle between $G_t(v)$ and $G_{t'}(v)$ is always less than $\pi/4$.

Let's start from the following model. Suppose we have a hypersurface in $\mathbb{R}^{n+1}$ with tangential data being the horizontal hyperplane. Let's construct the wrinkled embedding. First we introduce some notations.

Definition 4.2. *An oriented hypersurface $f : S \to \mathbb{R}^{n+1}$ is called:*

(1). ϵ-horizontal, if $\mathrm{G}\, df(S) \subset U_\epsilon(s) \subset S^n$;

(2). graphical, if $\mathrm{G}\, df(S) \subset U_{\pi/2}(s) \subset S^n$;

(3). ϵ-graphical, if $\mathrm{G}\, df(S) \subset U_{\pi/2+\epsilon}(s) \subset S^n$;

(4). quasi-graphical, if $\mathrm{G}\, df(S) \subset U_\pi(s) \subset S^n$.

Here s is the north pole of the sphere and $U_\delta(s)$ is the δ-neighbourhood of s where the distance is defined by arclength on S^n.

Lemma 4.2. *Let $f : S \to \mathbb{R}^{n+1}$ be an oriented quasigraphical hypersurface, such that S is ϵ-horizontal near ∂S. Then there exists a C^0-approximation of the embedding by an ϵ-horizontal wrinkled embedding $g : S \to \mathbb{R}^{n+1}$, such that g coincides with f near ∂S.*

Proof. Our first step is to fix some standard coordinates. Consider the height function $h = x_{n+1} \circ f : S \to \mathbb{R}$. f is already ϵ-horizontal near $\mathrm{Crit}(h)$. Hence by removing neighbourhoods of $\mathrm{Crit}(h)$, we may assume that h has no critical points on S. Now we will have a $(n-1)$-foliation by level sets $\{h^{-1}(t)\}_{t \in \mathbb{R}}$ and a 1-foliation by ∇h. This gives us standard coordinates on S by passing to an open cover of S by $\varphi_\alpha : D_\alpha \simeq D^{n-1} \times I$ where the first $(n-1)$-component correspond to the $(n-1)$-foliation and the last component correspond to the 1-foliation.

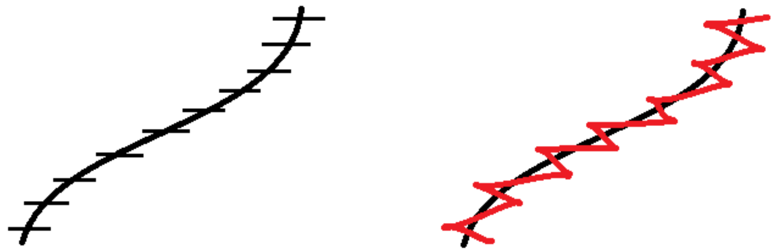

FIGURE 3. Constructing a wrinkled embedding from an embedding with a tangential rotation.

We start to construct a family of wrinkled embedding. Define a family of curves

$$x_t(u) = \frac{15}{8}\int_0^u (u^2-t)^2 dt, \;\; y_t(u) = -\frac{1}{2}(u^3 - 3tu).$$

This defines a continuous function $a_t(x)$ smooth away from $\pm\sqrt{t}$, where t is the parameter measuring how singular the curve is. Now to make the tangent vector sufficiently close to the vertical direction and make it periodic, we consider a family $A_{\sigma,t}$ of periodic-1 functions ($\sigma \in (0,1/8)$) so that

$$A_{\sigma,t} = \begin{cases} a_t\Big(\frac{x}{\sigma}\Big), & x \in \mathrm{Op}\,[-\sigma,\sigma], \\ 0, & x = \pm\frac{1}{2}. \end{cases} \quad \frac{dA_{\sigma,1}(x)}{dx} \in \begin{cases} [3,+\infty), & x \in (-\sigma,\sigma), \\ [-2,0), & x \in [2\sigma, 1/2-2\sigma], \\ (-\infty,-2], & x \in (\sigma, 2\sigma). \end{cases}$$

One can check that the composition $(t,u) \mapsto A_{\sigma,t}(x_t(u))$ is a smooth function. Now for sufficiently large $N \in \mathbb{N}$, let V be the horizontal unit vector transverse to S, and consider the flow

$$g(z,h) = \varphi_V^{\tau_0(z,h)}(z,h), \;\; \tau_0(z_1,...,z_{n-1},h) = A_{\sigma,t}\Big(\frac{2N+1}{2}h\Big)$$

will be the candidate for the wrinkled embedding on the single chart D_α.

To construct a wrinkled embedding on S, we need to glue the local models together. This will need two steps. The first one is to make the embedding smooth near the boundary of D_α, and the second step is to make the wrinkles in different charts D_α, D_β disjoint from each other.

For the first step, fix $\epsilon > 0$ and consider $\tilde{D}_\alpha = \varphi_\alpha(D^{n-1}_{1-2\epsilon} \times I_{1-1/N})$, where the integer N is to be determined. first we have to construct embryos to kill the wrinkles near the boundary. Let $\beta : [0,1] \to \mathbb{R}$ be a decreasing function such that

$$\beta(r) = \begin{cases} 1, & r \in [0, 1-3\epsilon], \\ 1-r-\epsilon, & r \in [1-2\epsilon, 1]. \end{cases}$$

Secondly we need to cut off the wrinkle near the boundary to make it flat. Thus we consider a cut-off function $\lambda : \mathbb{R}_{\geq 0} \to \mathbb{R}$ so that it equals 1 on $[0, 1-\rho]$ and is 0 near 1.Here $\rho = \min\{\epsilon/2, 1/2N\}$. Now let

$$\tau(z_1,...,z_{n-1},h) = \gamma\lambda(|z|)\lambda(|h|)A_{\sigma,\beta(|z|)}\Big(\frac{2N+1}{2}h\Big),$$

and $\tau_\alpha = \tau \circ \varphi_\alpha^{-1}$. For sufficiently large N and small γ, this defines the local wrinkled embedding.

For the second step. Consider $\mathbb{Q}\cap I$. Since the singularities all appear at rational values of h, we choose parametrizations φ_α so that $\varphi_\alpha(D^{n-1}\times(\mathbb{Q}\cap I))$ are pairwise disjoint. Yet this is not enough, we have to make the whole wrinkles pairwise disjoint. The wrinkles have width $8\sigma/(2N+1)$, so choose N sufficiently large such that

$$\varphi_\alpha\left(\bigcup_{k=-N}^{N}\left[\frac{2k-4\sigma}{2N+1},\frac{2k+4\sigma}{2N+1}\right]\right)$$

are pairwise disjoint. This completes the proof. $\square$

Proof of Theorem 4.1. Without loss of generality, let's assume $W=\mathbb{R}^q$. Fix a triangulation on V and assume that $\mathrm{Sing}(f_0)\subset \mathrm{sk}_{n-2}V$. Given a map $G:V\to \mathrm{Gr}_n\mathbb{R}^q$ define $G^*:V\to \mathrm{Gr}_n\mathbb{R}^q$ to be constant on each n-simplex with value defined by the barycenter of that simplex. It suffices to approximate G_t^* instead of G_t.

When $q=n+1$, first by the arguments in Theorem 3.7 we can construct a graphical isotopy $\tilde{f}_t:V\to\mathbb{R}^{n+1}$ with $\tilde{f}_0=f_0$, $\mathrm{G}\,d\tilde{f}_t$ is C^0-close to G_t on $\mathrm{Op}\,(\mathrm{sk}_{n-1}V)$. Now on each n-simplex, $d\tilde{f}_t$ is ϵ-graphical with respect to G_0, and hence by our assumption that the angle of tangential rotation is less than $\pi/4$, is quasi-graphical with respect to G_t. Now on each simplex the wrinkled embedding is constructed by the lemma, and they agree with the embedding on $\mathrm{sk}_{n-1}V$. Hence we've got a global wrinkled embedding.

When $q>n+1$, consider on each simplex the $n+1$-plane where the tangential rotation is happening and the conclusion follows from the previous case. This completes the proof. $\square$

5. Symplectic/Contact Structures

5.1. **Symplectic/Contact Structures on Open Manifolds.** Recall that for a ($2n$ dimensional) manifold V, a symplectic structure is a closed 2-form $\omega\in\Omega^2(V)$ such that $\omega^{\wedge n}\neq 0$. In this section we study the existence of symplectic and contact structures on open manifolds.

Definition 5.1. *For a manifold V, $\mathcal{R}_{symp}\subset\wedge^2V$ consist of the elements (x,β) such that $\beta^{\wedge n}\neq 0$. The space of almost symplectic forms is $S_{symp}=\mathrm{Sec}\mathcal{R}_{symp}$, and the space of genuine symplectic forms in the cohomology class $a\in H^2(V;\mathbb{R})$ is $\mathbb{S}^a_{symp}$.*

For an open manifold V and $\mathcal{R}\subset\wedge^pV$, denote by $\mathrm{Cl}_a\mathcal{R}$ be the closed p-forms in the cohomology class $a\in H^p(V;\mathbb{R})$. In general we have the following property.

Theorem 5.1. *Let V be an open manifold, $a\in H^p(V;\mathbb{R})$ and $\mathcal{R}\subset\wedge^pV$ be an open $\mathrm{Diff}\,V$-invariant relation. Then $\mathrm{Cl}_a\mathcal{R}\to\mathrm{Sec}\mathcal{R}$ is a homotopy equivalence.*

Proof. The theorem follows immediately from the h-principle for $\mathrm{Diff}\,V$-invariant open relations on open manifolds. The only thing that we would mention here is that the homology class is invariant under diffeotopy. $\square$

Corollary 5.2 (Gromov, 1969)**.** *Let V be an open manifold, $a\in H^p(V;\mathbb{R})$. Then $\mathbb{S}_{symp}\to S_{symp}$ is a homotopy equivalence. In particular, for any $a\in H^p(V;\mathbb{R})$ and any nondegenerate 2-form ω_0, there exists a homotopy through nondegenerate 2-forms such that $\omega_1\in a$ is symplectic.*

For a ($2n+1$ dimensional) manifold V, a (cooriented) contact structure is a (cooriented) codimension 1 distribution $\xi\subset TV$ being $\xi=\ker\alpha$ such that $\alpha\wedge d\alpha^{\wedge n}\neq 0$.

Definition 5.2. *For a manifold V, $\mathcal{R}_{cont}\subset\wedge^1V\oplus\wedge^2V$ consist of the elements (x,α,β) such that $\alpha\wedge\beta^{\wedge^n}\neq 0$. The space of almost contact structures S^+_{cont} is the conformal classes of $\mathrm{Sec}\mathcal{R}_{cont}$. The space of genuine contact structures is $\mathbb{S}^+_{cont}$.*

Also, for an open manifold V, $\mathcal{R} \subset \wedge^{p-1}V \oplus \wedge^p V$, $\mathrm{Ex}\mathcal{R} \subset \mathrm{Sec}\mathcal{R}$ is the subspace consisting of (α, β) such that $\beta = d\alpha$. In general we have the following property.

Theorem 5.3. *Let V be an open manifold and $\mathcal{R} \subset \wedge^{p-1}V \oplus \wedge^p V$ a $\mathrm{Diff}\, V$-invariant relation. Then $\mathrm{Ex}\mathcal{R} \to \mathrm{Sec}\mathcal{R}$ is a homotopy equivalence.*

Corollary 5.4 (Gromov, 1969)**.** *Let V be an open manifold. $S^+_{cont} \to \mathbb{S}^+_{cont}$ is a homotopy equivalence. In particular, given a nonvanishing 1-form α_0 and a nondegenerate 2-form β_0 on $\ker \alpha_0$, there is a homotopy such that α_1 is contact and $\beta_1 = d\alpha_1$.*

5.2. **Symplectic/Contact Structures on Closed Manifolds.** In general, the problems for symplectic/contact structures on closed manifolds are difficult and wildly open.

First let us talk about symplectic structures on closed manifolds. One necessary condition that can be deduced easily is that it must admit an nondegenerate 2-form and a nonzero $a \in H^2(V;\mathbb{R})$ (with $a^{\wedge n} \neq 0$). One may ask whether this is sufficient. In higher dimensions, it is still open, but in dimension 4 we can get the following negative answer.

Theorem 5.5 (Taubes, 1994)**.** *For $V = \#_{i=1}^{2k+1}\mathbb{CP}^2$, there exists a nondegenerate 2-form and a nonzero $a \in H^2(V;\mathbb{R})$ yet no symplectic structures.*

One may in addition consider the 1-parametric problem for symplectic structures. In this case, different from the previous problem, it is known that in dimension at least 6 the answer is false while it is open in dimension 4.

Theorem 5.6 (Ruan, 1994)**.** *For $n \geq 3$ there is a $2n$-dimensional manifold V such that there exist ω_0, ω_1 being symplectic forms on V that are homotopic through nondegenerate 2-forms but are not homotopic.*

Theorem 5.7 (McDuff, 1987)**.** *For $n \geq 3$ there is a $2n$-dimensional manifold V such that there exist ω_0, ω_1 being symplectic forms on V that are homotopic through nondegenerate 2-forms and $[\omega_0] = [\omega_1]$ but are not isotopic.*

We may also consider the extension problem of symplectic structures from $\mathrm{Op}\,\partial D^{2n}$ to D^{2n}. This is also related to the symplectic filling problem.

Theorem 5.8 (Gromov, 1985)**.** *There exists ω a symplectic form on $\mathrm{Op}\,\partial D^{2n}$ that extends to a nondegenerate 2-form on D^{2n} such that $\int_{D^{2n}} \omega^n > 0$, yet does not extend to a symplectic form on D^{2n}.*

Theorem 5.9 (Gromov, 1985)**.** *Let ω be a symplectic form on D^4 such that $\omega = \omega_{std}$ on $\mathrm{Op}\,\partial D^4$. Then there is a diffeomorphism $f : D^4 \to D^4$ being fixed near $\mathrm{Op}\,\partial D^4$ such that $f^*\omega = \omega_{std}$.*

However, we don't know whether the diffeomorphism f in the theorem can be chosen homotopic to the identity. We also don't know whether it is true in higher dimensions.

Next let us talk about contact structures on closed manifolds. For the h-principle of contact structures, we can show that it is π_0-surjective but not π_0-injective in dimension 3.

Theorem 5.10 (Martinet, 1971; Lutz, 1977)**.** *On any closed oriented 3-manifold V, any 2-plane field ξ is homotopic to a contact structure.*

Theorem 5.11 (Bennuquin, 1983)**.** *There exists an overtwisted contact structure ξ_{ot} on S^3 that is homotopic to ξ_{std} but not equivalent to ξ_{std}.*

For overtwisted contact structures on 3-manifolds, we have the following classification theorem. We also make a remark here that the classification theorem for overtwisted structures in higher dimensions has also been worked out recently.

Theorem 5.12 (Eliashberg, 1989). *Let V be an oriented manifold and*

$$\mathrm{Cont}_{ot}(V, D) \subset \mathbb{S}_{cont}$$

be the space of positive overtwisted contact structures on V which have D as an overtwisted disc and all coincide on $\mathrm{Op}\, D$. Let

$$\mathrm{Distr}(V, D) \subset S_{cont}$$

be the space of 2-plane fields η on V such that $\eta_{p_0} = T_{p_0}D$ for a fixed $p_0 \in D$. Then the inclusion

$$\mathrm{Cont}_{ot}(V, D) \to \mathrm{Distr}(V, D)$$

is a homotopy equivalence.

Just as we have considered in symplectic geometry, we can also consider the extension problem in contact geometry. In dimension 3 we have the following theorems. In higher dimensions, the problems are open.

Theorem 5.13. *Let α be a contact form on $\mathrm{Op}\, \partial D^3$ that formally extends to D^3. Then α extends to D^3 as a contact form. For overtwisted contact structures, this is also true; but for tight contact structures this is false.*

Theorem 5.14. *Let α be a tight contact form on D^3 such that $\alpha = \alpha_{std}$ on $\mathrm{Op}\, \partial D^3$. Suppose α is formally homotopic to α_{std} on D^3. Then there is a diffeoopy f_t on D^3 being fixed on $\mathrm{Op}\, \partial D^3$ such that $f_0 = \mathrm{id}, f_1^*\alpha = \alpha_{std}$.*

6. Overtwisted Contact Manifolds

7. Isotropic/Isosymplectic Embeddings

7.1. **Isosymplectic Embeddings into Symplectic Manifolds.** Let (W, ω_W) be a symplectic manifold. In this section if $f : V \to W$ is an embedding so that $f^*\omega_W$ defines a symplectic form on V, we call f a symplectic embedding; if in addition there is a priori a symplectic structure ω_V on V and $f^*\omega_W = \omega_V$, we will call it an isosymplectic embedding.

Theorem 7.1 (h-principle for isosymplectic embeddings; Gromov, 1986). *Let (V, ω_V) and (W, ω_W) be symplectic manifolds of dimension n and q respectively. $f_0 : V \to W$ is an embedding so that $f_0^*[\omega_W] = [\omega_V]$, and $F_0 = df_0$ is homotopic through monomorphisms F_s to an isosymplectic homomorphism $F_1 : TV \to TW$.*

(1). If $n \leq q - 2$ and V is open, then there is an isotopy $f_t : V \to W$ so that $f_1 : V \to W$ is isosymplectic and df_1 is homotopic to F_1 through isosymplectic monomorphisms.

(2). If $n \leq q - 4$ and V is closed, then there is a C^0-small isotopy $f_t : V \to W$ so that $f_1 : V \to W$ is isosymplectic and df_1 is homotopic to F_1 through isosymplectic monomorphisms.

The idea of the proof is as follows. When V is open, with the h-principle for directed embeddings, it is not difficult to get a symplectic embedding. However we still have to make $f^*\omega_W$ strictly agree with ω_V. To reach this goal, what we do is to consider a standard neighbourhood of the symplectic neigbourhood of $f(V)$ which is isomorphic to a symplectic vector bundle over V, and perturb it along the fiber of the vector bundle. When V is closed, we just apply the microextension trick.

Proof in the open case. First since $A_{\mathrm{symp}} \subset \mathrm{Gr}_n W$ is open, according to the h-principle for directed embeddings we will get an isotopy $\tilde{f}_t : V \to W$ such that $\tilde{f}_0 = f_0$, $\tilde{f}_1$ is symplectic and $d\tilde{f}_1$ is homotopic to F_1. Hence there is a homotopy of nondegenerate 2-forms connecting $\tilde{f}_1^*\omega_W$ and $F_1^*\omega_W$. Now since $[\omega_V] = [f_0^*\Omega_W]$ we know that there is a homotopy ω_t of symplectic forms so that

$$\omega_0 = \tilde{f}_1^*\omega_W,\ \omega_1 = \omega_V,\ [\omega_t] \equiv [\omega_V].$$

Thus we can write $\omega_t = \omega_0 + d\alpha_t$. It suffices to show the following proposition. $\square$

Proposition 7.2. *Let V be a manifold of dimension n and $f : V \to (W, \omega_W)$ be a symplectic embedding where W has dimension $q > n$. Set $\omega_0 = h_0^*\omega_W$. Let $\omega_t = \omega_0 + d\alpha_t$ be a homotopy of symplectic forms. Then there is an arbitrary C^0-small symplectic isotopy h_t such that $h_1^*\omega_W = \omega_1$.*

Proof. We try to approximate the homotopy α_t be a piecewise primitive homotopy $\tilde{\alpha}_t$. Here a 1-form α is called primitive if $\alpha = rds$ for $r, s \in C_c^\infty(V)$. This is the following lemma.

Lemma 7.3. *Let α_t be a family of 1-forms on V. Then there is a piecewise primitive homotopy $\tilde{\alpha}_t$ with $\tilde{\alpha}_0 = \alpha_0, \tilde{\alpha}_1 = \alpha_1$, such that*

$$\|\alpha_t - \tilde{\alpha}_t\|_{C^1} \leq \epsilon,\ t \in [0, 1].$$

Proof. Note that any homotopy α_t can be C^1-approximated by a piecewise linear homotopy. Hence we may assume that $\alpha_t = t\alpha$. Furthermore, as long as we have constructed a piecewise primitive homotopy β_t so that $\beta_0 = 0, \beta_1 = \alpha$, then for $N \in \mathbb{N}$ sufficiently large, we can set

$$\tilde{\alpha}_{j/N+\tau} = (j/N)\alpha + (1/N)\beta_{N\tau}.$$

Then this will be C^1-close to α_t.

Consider a finite partition of unity on V and on a local chart U_j write

$$\rho_{U_j}\alpha = r_1 dx_{U_j,1} + ... + r_n dx_{U_j,n}$$

where $\operatorname{supp} r_i \subset\subset U$. Take another cutoff function θ_{U_j} on U_j such that $\theta_{U_j}|_{\operatorname{supp} r_i} \equiv 1$. Then

$$\rho_{U_j}\alpha = r_1 d(\theta_{U_j} x_{U_j,1}) + ... + r_n d(\theta_{U_j} x_{U_j,n}).$$

This defines the piecewise primitive homotopy. $\square$

Now we go back to the proof of the proposition. Given the piecewise primitive homotopy of symplectic forms, the goal is to realize the homotopy by an isotopy $f_t : V \to W$. Without loss of generality, we only need to construct the isotopy for $\omega_0 + tdr \wedge ds$ where $\operatorname{supp} r, \operatorname{supp} s$ are in a coordinate chart $U \subset V$. Consider a symplectic neighbourhood of U being

$$(U \times D_\epsilon^2 \times D_\epsilon^{q-n-2}, \omega_0 + \omega_{\text{std},2} + \omega_{\text{std},n-q-2}) \longrightarrow (W, \omega_W).$$

Consider the map $\phi = (r, s) : U \to D_r^2$ and compose it with an area preserving map $\pi : D_r^2 \to D_\epsilon^2$. Then define the isotopy by

$$g_t(x) = \begin{cases} t\pi \circ \phi(x), & x \in U; \\ x, & x \notin U. \end{cases}$$

This completes the proof of the proposition. $\square$

Proof in the closed case. Consider the $(n-1)$-skeleton $K = \operatorname{sk}_{n-1}V$ of some triangulation of V and apply the theorem to $\operatorname{Op} K$. Then we will get an isosymplectic embedding near K. Now we only need to show how to extend the embedding from $\partial\Delta^n$ to Δ^n. We apply the microextension trick to consider the open manifold $\Delta^n \times D_\epsilon^2$. Suppose

$$[\omega_V]_{H^2(\Delta^n, \partial\Delta^n)} = [f_0^*\omega_W]_{H^2(\Delta^n, \partial\Delta^n)}.$$

Then the relative version of the h-principle gives us the embedding.

The relative cohomology condition is automatically true when $n > 2$. When $n = 2$, we have to show that there is an isotopy f_t so that for any 2-simplex Δ^2, $f_t|_{\partial\Delta^2} = f_0$ and

$$\int_{\Delta^2} f_1^*\omega_W = \int_{\Delta^2} \omega_V.$$

Globally we know that $f_0^*\omega_W - \omega_V = \alpha$. Fix any 1-simplex σ we can define

$$A_\sigma = \int_\sigma \alpha.$$

Take an integer N of the same sign as A_σ, and $\theta : [0,1] \to [0, 2\pi N]$ so that $\theta = 0$ near 0 and $\theta = 2\pi N$ near 1. Now for sufficiently small $\epsilon > 0$ let

$$h_t^\sigma : I \to \operatorname{Op} I, \ \ h_t^\sigma(x_1) = (x_1, 0, t\epsilon(1-\cos\theta(x_1)), t\epsilon \sin\theta(x_1), 0, ..., 0).$$

Note that h_t^σ is fixed near boundary and

$$\int_{h_t^\sigma(I)} \lambda_{\text{std}} = \pi N \epsilon^2 t.$$

Let $\epsilon = \sqrt{A_\sigma/N\pi}$ then we have

$$\int_{h_t^\sigma(I)} \lambda_{\text{std}} = t A_\sigma.$$

Let N be large enough such that h_t^σ is C^0-close to the original inclusion. Now let $f_t|_\sigma = h_f^\sigma \circ f_0|_\sigma$. Now we know that by Stokes formula

$$\int_{\Delta^2} f_1^*\omega_W = \int_{\Delta^2} \omega_V.$$

This completes the proof. $\square$

7.2. **Isocontact Embeddings into Contact Manifolds.** Let (W, ξ_W) be a contact manifold. In this section if $f : V \to W$ is an embedding so that $f_*\xi_V$ defines a contact structure on W, we call f a symplectic embedding; if in addition there is a priori a symplectic structure ξ_V on V and $f^*\xi_V = \xi_W$, we will call it an isosymplectic embedding.

Theorem 7.4 (h-principle for isocontact embeddings; Gromov, 1986)**.** *Let (V, ξ_V) and (W, ξ_W) be contact manifolds of dimension n and q respectively. $f_0 : V \to W$ is an embedding so that $F_0 = df_0$ is homotopic through monomorphisms F_s to an isocontact homomorphism $F_1 : TV \to TW$.*

(1). If $n \le q-2$ and V is open, then there is an isotopy $f_t : V \to W$ so that $f_1 : V \to W$ isisosymplectic and df_1 is homotopic to F_1 through isosymplectic monomorphisms.

(2). If $n \le q-4$ and V is closed, then there is a C^0-small isotopy $f_t : V \to W$ so that $f_1 : V \to W$ is isocontact and df_1 is homotopic to F_1 through isocontact monomorphisms.

As in the case of isosymplectic embeddings, when V is an open manifold we can first get a contact embedding by the h-principle for directed embeddings. Then we try to consider a standard neighbourhood of the contact neigbourhood of $f(V)$ which is isomorphic to a symplectic vector bundle over V, and perturb it along the fiber of the vector bundle. When V is closed, we just apply the microextension trick. Therefore, the theorem can also be reduced to the following proposition.

Proposition 7.5. *Let V be a manifold of dimension n and $f : V \to (W, \omega_W)$ be a symplectic embedding where W has dimension $q > n$. Set $\omega_0 = h_0^*\omega_W$. Let $\omega_t = \omega_0 + d\alpha_t$ be a homotopy of symplectic forms. Then there is an arbitrary C^0-small symplectic isotopy h_t such that $h_1^*\omega_W = \omega_1$.*

However, unlike the case of isosymplectic embeddings where we consider a multiple covering $\pi : D_r^2 \to D_\epsilon^2$, here in order to compress the contact tubular neighbourhood, we will introduce the notion of contactly contractible domains.

Definition 7.1. *Let $U \subset (V, \xi_V)$ be a compact domain with piecewise smooth boundary. U is contactly contractible if there is a contact vector field X on V that is transverse to ∂U and the flow φ_t^X satisfies $(\varphi_t^X)^*\alpha \to 0\,(t \to \infty)$.*

Lemma 7.6. *Let ξ_t be a family of contact structures and α_t be a family of corresponding contact forms. Then there is a sequence of primitive 1-forms $\beta_i = r_i ds_i\,(1 \le i \le M)$ such that*

(1). $\alpha_1 = \alpha_0 + \sum_{i=1}^M \beta_i$;

(2). $\tilde{\alpha}_{k/M} = \alpha_0 + \sum_{i=1}^k \beta_i$ is contact for any $0 \le k \le M$;

(3). for any $1 \le i \le n$, r_i, s_i are supported in a contactly contractible domain homeomorphic to a ball with respect to the contact form $\alpha_{i/M}$.

Proof. First for any $t \in [0,1]$ we cover V be domains U_t^i that is isomorphic to either the ball $B_r(0)$ or the semiball $B_r(0) \cap \{x | l(x) \ge 0, l \in (\mathbb{R}^n)^*\}$ inside

$$\left(\mathbb{R}^n,\ \xi_{\mathrm{std}} = \ker\left(dz - \sum_{i=1}^m (x_i dy_i - y_i dx_i)\right)\right).$$

These are all contactly contractible domains. By continuity there exists $\delta > 0$ such that all $U_t^i\,(1 \le i \le M)$ are compactly contractible for all $\tau \in [t-\delta, t+\delta]$. Now consider a partition of unity for each $t \in [0,1]$ and consider

$$\alpha_{ij} = \alpha_{i/N} + \sum_{k=1}^j \rho_{U^k_{i/N}}(\alpha_{(i+1)/N} - \alpha_{i/N}).$$

Now as in the symplectic case we can approximate these 1-forms by piecewise primitive 1-forms. This completes the proof. □

Proof in the open case. Suppose we have already got a contact embedding that is not necessarily isocontact. We now write $\alpha_1 = \alpha_0 + \sum_{i=1}^M \beta_i$ and construct inductively isocontact embeddings

$$f_j : \big(V, \ker(\tilde{\alpha}_{j/M})\big) \to (W, \xi_W).$$

Suppose f_j has already been constructed. Now $\tilde{\alpha}_{j/M}, \tilde{\alpha}_{(j+1)/M}$ only differ over a contactly contractible ball U_j with respect to the contact structure $\tilde{\alpha}_{j/M}$. Consider the contact tubular neighbourhood

$$F_j : \big(U \times D_\epsilon^2 \times D_\epsilon^{q-n-2}, \ker\big(\tilde{\alpha}_{j/M} + \lambda_{\mathrm{std},2} + \lambda_{\mathrm{std},q-n-2}\big)\big) \to (W, \xi_W).$$

Suppose $\tilde{\alpha}_{j/M} - \tilde{\alpha}_{(j+1)/M} = r ds$. Then we may set

$$F : U \to U \times \mathbb{R}^2,\ \ F(u) = (u, r(u), s(u)).$$

This is an isocontact embedding. In order to compress it to an embedding $\tilde{F} : U \to U \times D_\epsilon^2$, we use the fact that U is contactly contractible.

Let $X = X_H$ be the contact vector field so that $(\varphi_t^X)^*\tilde{\alpha}_{j/M} = h_t \tilde{\alpha}_{j/M}$. Let $\tilde{U} \subset U$ be a domain with piecewise smooth boundary, X points inward to $\tilde{U}$ transversely and $\operatorname{supp} r, \operatorname{supp} s \subset \tilde{U}$. Take a cutoff function θ so that $\varphi|_{\tilde{U}} \equiv 1, \varphi|_{\mathrm{Op}\,\partial U} = 0$. Let $\tilde{X} = X_{\theta H}$. Then

$$\big(\varphi_t^{\tilde{X}}\big)^*\alpha = \tilde{h}_t \alpha,\ \ \tilde{h}_t|_{\tilde{U}} = h_t|_{\tilde{U}} \to 0,\ \tilde{h}_t|_{\mathrm{Op}\,\partial U} \equiv 1.$$

Now the following map gives a contactomorphism

$$\psi_t(u,x,y) = \Big(\varphi_t^{\tilde{X}}(u), \big(\tilde{h}_t\big)^{1/2} x, \big(\tilde{h}_t\big)^{1/2} y\Big).$$

Then we consider the composition

$$g_t = \pi \circ \psi_t \circ F : U \to \mathbb{R}^2.$$

When $u \in \tilde{U}$, $g_t(u) = h_t(u)g_0(u) \to 0$; when $u \in U \backslash \tilde{U}$, $g_0(u) = (r(u), s(u)) = 0$ so we still have $g_t(u) = 0$. Therefore we can make

$$(\psi_t \circ F)(U) \subset U \times D^2_\epsilon$$

when t is sufficiently large and $\psi_t \circ F|_{\mathrm{Op}\,\partial U} \equiv \mathrm{id}$. This gives the required isocontact embedding, which completes the proof of the theorem. □

The proof of the closed case is even simpler than that for isosymplectic embeddings since the arguments work without any cohomological assumptions. Therefore we omit the proof.

7.3. **Isotropic Embeddings into Symplectic/Contact Manifolds.** The h-principle for subcritical isotropic embeddings is a direct corollary of the previous theorems.

Definition 7.2. *Let W be a symplectic/contact manifold. $\mathbb{I}\mathrm{sot}_{emb}(V, W)$ is the space of isotropic embeddings from V to W, and $\mathrm{Isot}_{emb}(V, W)$ is the space of embeddings with an isotropic monomorphism covering the embedding.*

Theorem 7.7 (h-principle for subcritical isotropic embeddings)**.** *Let $\dim V = n, \dim W = q$. If $n < [(q-1)/2]$, then $\mathbb{I}\mathrm{sot}_{emb}(V, W) \to \mathrm{Isot}_{emb}(V, W)$ is a homotopy equivalence.*

Proof. The idea is to extend a (formal) isotropic embedding to a (formal) isosymplectic/isocontact embedding and then apply the previous results. Given a formal isotropic embedding, one may want to extend the embedding to T^*V in the symplectic case or $J^1(V)$ in the contact case. Consider $F_s : TV \to TW$. F_1 is isotropic means that it can be extended to a symplectic monomorphism

$$\tilde{F}_1 : TV \oplus T^*V \to TW$$

in the symplectic case (or a contact monomorphism

$$\tilde{F}_1 : TV \oplus T^*V \oplus \mathbb{R} \to TW$$

in the contact case). Now we want to lift the homotopy F_s to a homotopy $\tilde{F}_s$ so as to give an extension of the embedding $f : V \to W$ to $\tilde{f} : T^*V \to W$ (or $J^1(V) \to W$) by the exponential map. In fact consider $f^*W/F_t(TV)$, this is equivalent to extend a subbundle $\tilde{F}_t(TV \oplus T^*V)/F_t(TV)$ (or $\tilde{F}_t(TV \oplus T^*V \oplus \mathbb{R})/F_t(TV)$) on V to a subbundle on $V \times I$. Since $V \simeq V \times I$, such an extension always exists.

Now the extended embeddings are isosymplectic or isocontact and the dimensions are subcritical. Then the theorem follows from the h-principle for isosymplectic/isocontact embeddings. □

8. Legendrian/Lagrangian Immersions

8.1. **Integrability, Microflexibility and Applications.** As we have promised before, we first show that isotropic/isosymplectic/isocontact immersions into symplectic/contact submanifolds are locally integrable and microflexible, and then give some direct applications.

Lemma 8.1. *The differential relation defining isotropic/isosymplectic/isocontact immersions into symplectic/contact manifolds are locally integrable.*

Proof. This is basically Moser's theorem and Gray's stability theorem. Let's consider isosymplectic immersions. Given any point $v \in V$ and $F(v) \in \mathcal{R}_{\mathrm{isosymp}}$, we have a symplectic monomorphism on the tangent space. By Moser's theorem, one can use the exponential map to extend F to a holonomic section on a neighbourhood of v. For isocontact immersions, given any point $v \in V$ and $F(v) \in \mathcal{R}_{\mathrm{isocont}}$, we have a symplectic monomorphism on the tangent space restricting to the hypersurface distribution. By Moser's theorem, one can again use the exponential map to extend F to a neighbourhood of v.

For isotropic embeddings, one has an isotropic monomorphism $T_vV \to T_wW$. Then one can canonically determine a symplectic monomorphism $T_vV \oplus T_v^*V \to T_wW$ (one still has to restrict to the hyperplane distribution when dealing with the contact case). Then by Moser's theorem, one can again use the exponential map to extend F to a neighbourhood of v. □

Lemma 8.2. *(1). The differential relations defining isotropic/isosymplectic immersions into symplectic manifolds is k-microflexible for $k \neq 1$;*

(2). The differential relations defining isotropic/isocontact immersions into symplectic manifolds is microflexible.

Proof. Our goal is to extend a isotropic/isosymplectic/isocontact isotopy from $\mathrm{Op}\,(\partial I^n \cup I^k)$ to $\mathrm{Op}\,I^n$. By isotopy extension theorem in symplectic/contact geometry, the only obstruction for isotropic/isosymplectic/isocontact isotopy is generated by a flow on the abient symplectic/contact manifolds is $H^2(I^n, \partial I^n \cup I^k; \mathbb{R})$ in the symplectic case.

In fact, first let's choose an arbitrary extension ψ_t of the homotopy. In general this is not symplectic/contact where the difference is captured by

$$\alpha_t = \frac{d}{dt}\left(\psi_t^*\omega\right).$$

Now it suffices to use Moser's theorem / Gray's theorem to adjust the symplectic/contact structure. In the contact case, one can always find a contact Hamiltonian to generate the isotopy. In the symplectic case, this is not always true. However, when $H^2(I^n, \partial I^n \cup I^k; \mathbb{R}) = 0$, which means α_t is exact, one can find a symplectic flow to generate the isotopy. This shows the claim. □

Lemma 8.3. *The Hamiltonian diffeomorphism groups/contactomorphism groups are capacious.*

Proof. This comes from the fact that any smooth function will give you a group element and one can cut-off smooth functions, choose differentials of smooth functions arbitrarily. □

Now we consider immersions into contact manifolds. The first theorem is almost a direct corollary from Section 3.5 and the microextension trick.

Theorem 8.4 (h-Principle for Isocontact/Isontropic Immersions; Gromov)**.** *If* $\dim V < \dim W$*, then all forms of h-principles hold for isocontact immersions; all forms of h-principles hold for isotropic or Legendrian immersions.*

Theorem 8.5 (h-Principle for Transverse Immersions into Contact Manifolds)**.** *Let* (W, ξ) *be a contact manifold and* $\dim V < \dim W$*. Then immersions $f : V \to W$ transverse to ξ satisfy all forms of h-principles.*

Proof. Again we use the microextension trick. Let

$$\mathcal{R} = \mathcal{R}_{\mathrm{imm}} \cap \mathcal{R}_{\mathrm{trans}} \subset J^1(V, W).$$

Now consider $\tilde{\mathcal{R}} \subset J^1(V \times \mathbb{R}, W)$ be the immersions transverse to ξ but tangent along each fiber $\{v\} \times \mathbb{R}$. The relation of transverse immersions is open, and the relation of being tangent to fibers is locally integrable and microflexible. In addition the relation is invariant under diffeomorphisms like

$$(x, t) \mapsto (x, h(x, t)).$$

Thus h-principles hold. Now by restricting to the zero section we get a transverse immersion. □

8.2. **Lagrangian and Isosymplectic Immersions.** To get the h-principle for Lagrangian /isosymplectic immersions, the main obstruction is that the relation is not 1-microflexible. In this subsection we solve this problem.

Theorem 8.6 (h-Principle for Lagrangian Immersions; Gromov, 1971)**.** *All forms of h-principles hold for Lagrangian immersions $f : V \to (W, \omega)$ as long as $[f^*\omega] = 0$ is required for formal Lagrangian immersions.*

Proof. First we prove that all h-principles hold for exact Lagrangian immersions. In fact after lifting to the contactization $W \times \mathbb{R}$ we will know the result from Theorem 8.4.

Now consider the general case. First by Theorem 3.6 we can assume that $f : V \to W$ is already an immersion. Consider the map $\tilde{f} : \nu V \to W$. Now by assumption $[\tilde{f}^*\omega] = 0$. Hence $(\nu V, \tilde{f}^*\omega)$ is an exact symplectic manifold. Now the result follows from the exact case.

Next we consider the parametric case. By the same argument we have a family of immersions $f_t : V \to W$, $t \in I^k$ that is holonomic along $\mathrm{Op}\,(\partial I^k)$. Let $\tilde{f}_t : \nu V \to W$ be the lifting. In order to apply the h-principle for the exact case, we not only need the immersions $f_t : V \to W$, $t \in \mathrm{Op}\,(\partial I^k)$ to be Lagrangian, but exact Lagrangian in fact. To do this let $d\theta_t = \tilde{f}_t^*\omega$. Choose $\tilde{\theta}_t$ on νV so that

$$\tilde{\theta}_t = \pi^*(\theta_t|_V),\ t \in \partial I^k.$$

Now we consider the family of forms $\{\theta_t - \tilde{\theta}_t\}_{t \in I^k}$. Now the Lagrangian immersions $f_t : V \to \nu V$, $t \in \partial I^k$ becomes exact. ☐

Theorem 8.7 (h-Principle for Isosymplectic Immersions; Gromov, 1971)**.** *If $\dim V < \dim W$, then all forms of h-principles hold for isosymplectic immersions $f : (V, \omega_V) \to (W, \omega_W)$ as along as $[f^*\omega_W] = [\omega_V]$ is required for formal isosymplectic immersions.*

Proof. By the microextension trick, it suffices to show the local h-principle. The key idea is to realize the isosymplectic embedding by its graph in $(V \times W, \omega_V \times (-\omega_W))$ which is isotropic.

Still we first consider the exact case where $\omega_V \times (-\omega_W)$ is exact on some subset $U \subset V \times W$ and show that isotropic sections $V \to V \times W$ satisfy local h-principles. This is easily shown by passing to sections in the contactization.

Now consider the general case. Let F be a formal solution with $\mathrm{bs}\,F = f$. Choose a neighbourhood U of the graph of f in $V \times W$. Now the assumption

$$[f^*\omega_W] = [\omega_V]$$

tells us that $\omega_V \times (-\omega_W)$ is exact on U. Then the proof follows from the exact case. ☐

9. Loose Legendrian Embeddings

9.1. **Legendrians in 3-Dimensional Manifolds.** First to better understand Legendrian submanifolds, we introduce the notion of a front porjection. Consider the contact manifold $J^1(V)$, the front projection is defined by

$$\pi = \pi_{\text{front}} : J^1(V) \to J^0(V) = V \times \mathbb{R}.$$

Locally on a coordinate chart $(U; x, y, z)$, the front diagram uniquely determines the Legendrian by

$$y_i = \frac{\partial z}{\partial x_i}.$$

In dimension 3, the front diagram is smooth whenever the Legendrian is not tangent to the fiber $T_v^*V \times \{t\}$. For the points where Legendrian is tangent to the fiber, a cusp will appear in the front diagram.

FIGURE 4. On the left is a formal Legendrian embedding in the front projection, the short straight lines correspond to the tangential data; on the right is the approximation by a genuine Legendrian embedding in the front projection.

FIGURE 5. Stablization of a Legendrian curve.

Now consider a formal Legendrian curve in $\mathbb{R}^3$. This gives a curve on $\mathbb{R}^2$ and a (possibly singular) tangent line field. To approximate the formal Legendrian by a genuine Legendrian, it suffices to construct a cuspital curve on $\mathbb{R}^3$. Definitely without a cusp in general this cannot be done. However one can do this by approximating the formal one by zig-zags. In dimension 3, this is called stablization of Legendrian curves.

Definition 9.1. *Given a Legendrian curve Λ, choose a standard local chart U so that the front projection $\pi(\Lambda \cap U) = \{(x, 0) | x \in \mathbb{R}\}$. The stablization of Λ is a Legendrian obtained by replacing $\pi(\Lambda \cap U)$ by a zig-zag that coincides with the x-axis near the boundary.*

We should make a remark here that stablization changes the formal Legendrian class. For a Legendrian curve Λ defined by $f_\Lambda : S^1 \to \mathbb{R}^3$. Then $\xi|_\Lambda$ is a trivial 2-plane field. Consider the Gauss map

$$\mathrm{G}\, df_\Lambda : S^1 \to \xi|_\Lambda.$$

Under the trivialization of $\xi|_\Lambda$, the degree of the map $\mathrm{G}\, df_\Lambda : S^1 \to \mathbb{R}^2 \backslash \{0\}$ is called the rotation number. This is a formal invariant of Legendrian curves. However, stablization will increase the rotation number by 1 (see, for example, Geige's *An Introduction to Contact Topology*, Proposition 3.5.19).

However, the previous zig-zag argument still tells us that after stablizing one can approximate any formal Legendrian by a genuine Legendrian.

Theorem 9.1 (Fuchs-Tabachnikov, 1997)**.** *Let $f_t : S^1 \to (Y, \xi)$ be a family of Legendrian embeddings, with $t \in S^{k-1}$, so that the family $\{f_t\}$ extends to D^k among formal Legendrian embeddings. Then after stabilizing the Legendrians f_t sufficiently many times (with both orientations), the family extends to D^k among genuine Legendrian embeddings.*

9.2. **Loose Legendrians in Higher Dimensional Manifolds.** As we saw in the previous section, when working in the front projection, to get a Legendrian embedding it seems necessary to allow cusp singularities and consider their isotopies. Hence we may consider to apply the wrinkled embedding theorem, which deals with directed embeddings with cusp singularities.

Definition 9.2. *Let* $\dim V = n$, $\dim W = 2n+1$ *and* ξ *be a contact structure on* W. *A wrinkled Legendrian embedding is a smooth map* $f : V \to (W, \xi)$ *which is a topological embedding such that*

(1). $df(TV) \subset \xi$ *and* df *has full rank outside a codimension 2 subset in* V*;*

(2). The singularities where df *is not of full rank is diffeomorphic to* $\sqcup_{j\in I} S_j^{n-2}$, *where* S_j^{n-2} $(j \in I)$ *are called Legendrian wrinkles;*

(3). Each S_j^{n-2} *is contained in a Darboux chart* U_j *so that* $U_j \cap V$ *is diffeomorphic to* $\mathbb{R}^n$, *and the front projection* $\pi_j \circ f : V \cap U_j \to \mathbb{R}^{n+1}$ *is a wrinkled embedding.*

A parametric family of Legendrian embedding is a smooth family $f_t : V \to (W, \xi)$ *of wrinkled Legendrians that allow Legendrian lifts from front projection of embryo singularities, called Legendrian embryos.*

Here we are imposing some strong conditions that each wrinkle is contained in some Darboux chart. However, note that the Darboux charts can intersect.

Proposition 9.2. *Let* $(f_t, F_{s,t})$ *be a family of formal Legendrian embeddings* $V \to (W, \xi)$ *for* $t \in I^k$. *Then the family is homotopic through formal Legendrian embeddings to a wrinkled Legendrian embedding* $\tilde{f}_t : V \to (W, \xi)$.

To prove this theorem, it suffices to give any formal Legendrian embedding into a contact manifold a well-defined front projection. This means we should make any formal Lgenedrian embedding graphical. The following proposition is nontrivial because we have to ensure that the map from the neighbourhood to $J^1(V)$ preserves contact structure.

Proposition 9.3. *Let* (f, F_s) *be a formal Legendrian embedding* $V \to (W, \xi)$. *Then after a small isotopy from* f *to* $\tilde{f}$, *there is an open neighbourhood of* $f(V)$ *that is contactomorphic to an open subset in* $J^1(V)$ *and* $f(V)$ *is a section.*

For wrinkled Legendrians, the cusp singularities of the front projection will not affect smoothness, but the singularities where birth-death of cusps appear are affecting smoothness (they are the real singularities). To get an h-principle for real Legendrian embeddings instead of wrinkled Legendrian embeddings, we also need to resolve the singularities that appear in wrinkled Legendrian embeddings. First let's give a local model for resolving Legendrian wrinkles.

Definition 9.3. *Let* $f : V \to (W, \xi)$ *be a wrinkled Legendrian embedding. A marking for* f *is a codimension 1 embedded submanifold* Φ *so that* $\partial\Phi$ *is mapped diffeomorphically to the Legendrian wrinkles,*

$$\pi_j \circ f(\Phi \cap U_j) = \{(x, z) \in \mathbb{R}^n \times \mathbb{R} | z = 0, |x| \geq 0\},$$

and the interior of Φ *is disjoint from the Legendrian wrinkles.*

Let $f_t : V \to (W, \xi)$ *be a family of wrinkled Legendrian embeddings. A family of markings* Φ_t *is required to vary smoothly in* $t \in I^k$ *away from where Legendrian embryos appear. When embryos appear, then for some parameter* $s \in I$ *so that* I *is transverse to the (codimension 1) embryo singularity and intersecting at* $s = 0$,

$$\pi_j \circ f(\Phi_s \cap U_j) = \{(x, z) \in \mathbb{R}^n \times \mathbb{R} | z = 0, |x|^2 \geq s\}, \ \ s \in (-\epsilon, \epsilon).$$

Lemma 9.4. *Let* $f_t : V \to (W, \xi)$ *be a family of wrinkled Legendrian embeddings, and* Φ_t *a family of markings. Then there is a* C^0*-close wrinkled Legendrian embedding* $\tilde{f}_t : V \to (W, \xi)$ *that is smooth on a neighbourhood of* Φ_t *and coincides with* f_t *outside that neighbourhood.*

Figure 6. Resolving the singularity near the Legendrian embryo.

Before proving the lemma, let's fix some notations. Let $\delta > 0$. $m_\delta(x) : \mathbb{R} \to [0,1]$ be a smooth function so that $m_\delta(x) = x$ for $x \geq 2\delta$, $m_\delta(x) = \delta$ for $x \leq \delta/2$, and $0 \leq m_\delta'(x) \leq 1$ everywhere. This will also be used in Lemma 9.5.

Proof. We work on a standard local model. For $u \in \mathbb{R}$, write

$$\psi_r(u) = \left(u^3 - ru, \frac{9}{4}u^5 - \frac{5r}{2}u^3 + \frac{5r^2}{4}u\right).$$

Then near the boundary of Φ, let r be the radius coordinate and u be the transverse coordinate (with respect to Φ), we have

$$\pi_j \circ f : V \cap U_j \to \mathbb{R}^{n+1};\ \ (\theta, r, u) \mapsto (\theta, r, \psi_r(u)).$$

Now we define $\tilde{f}$ to be the map determined by

$$\pi_j \circ f : V \cap U_j \to \mathbb{R}^{n+1};\ \ (\theta, r, u) \mapsto (\theta, r, \psi_{m_\delta(r)}(u))$$

for $\delta > 0$ sufficiently small. This completes the proof. $\square$

Now we are able to construct smooth Legendrians from wrinkled Legendrians as long as we can define a (family of) marking(s). However, notice that in parametric families, markings can only change from a disk to an annuli or vice versa, and can never disappear.

Hence we would try to pair each Legendrian embryo with neighbourhoods of resolved singularities through an annuli marking at the beginning, and then let the marking to change to a disk in the end, obtaining only neighbourhoods of resolved singularities. This means that in order to get an h-principle, one will have to find arbitrarily many neighbourhoods of resolved singularities. This leads to the definition of loose Legendrians.

Definition 9.4. *Let $I^3 \subset \mathbb{R}^3$ be a cube of side length 1, $V_0 \subset I^3$ be a Legendrian curve whose front projection is a zig-zag and is equal to $\{(x,y,z) | y = z = 0\}$ near ∂I^3. Let $\rho > 1$. $D_\rho = \{(q,p) \in \mathbb{R}^{2(n-1)} | |p| \leq \rho, |q| \leq \rho\}$. $Z_\rho = \{(q,p) | p = 0, |q| \leq \rho\}$. Then a standard loose chart is*

$$(I^3 \times D_\rho, V_0 \times Z_\rho).$$

Let $V \to (W, \xi)$ be a Legendrian submanifold. Then a loose chart is an open subset $U \subset W$ so that

$$(U, V \cap U) \simeq (I^3 \times D_\rho, V_0 \times Z_\rho).$$

The Legendrian embedding is loose if there exists a loose chart.

Lemma 9.5. *A loose chart of size $\rho > 1$ contains arbitrarily many loose charts of the same size $\rho > 1$.*

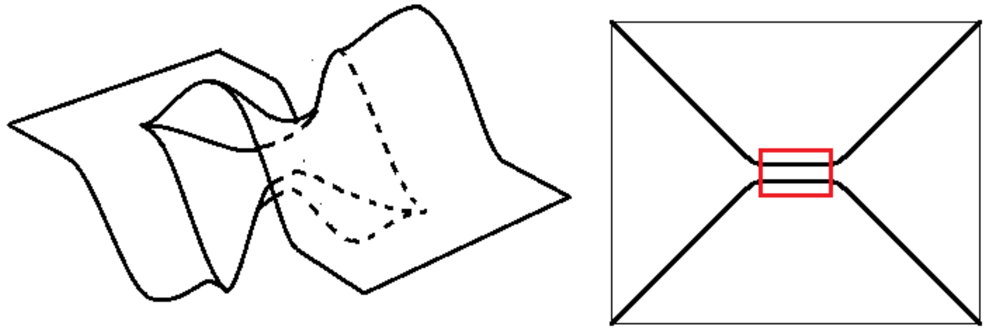

FIGURE 7. Shrinking the loose chart to get a long tube containing arbitrarily many loose charts. The red region contains many loose charts.

Proof. Given the standard zig-zag V_0, define a Legendrian V by the following front

$$\{(q,x,z)|(x,z)\in m_\delta(|q|+1-\rho)V_0\}.$$

Since $m'_\delta(x)\le 1$ we know that the Legendrian $V\subset I^3\times D_\rho$. In addition since V is Legendrian isotopic to $V_0\times Z_\rho$ rel boundary, it is also contact isotopic to $V_0\times Z_\rho$ rel boundary. Now we only need to construct two distinct loose charts of V.

Consider $\{(q,x,z)||q|\le\tilde\rho\}$ where $\tilde\rho=\rho-1-\delta/2$. On this subset $V=\delta V_0\times Z_{\tilde\rho}$. By scaling we can see that this is contactomorphic to $(V_0\times Z_{\tilde\rho/\delta}, I^3\times D_{\tilde\rho/\delta})$. Now choose $\delta>0$ small enough and we will get enough loose charts. □

Now we are able to prove the h-principle for loose Legendrian embeddings.

Theorem 9.6 (h-Principle for Loose Legendrians; Murphy, 2012)**.** *For any family of formal Legendrian embeddings $(f_t,F_{s,t})$ mapping $V\to(W,\xi)$ (where $t\in I^k$) that is genuine Legendrian near $t\in\partial I^k$ and on a fixed loose chart $(U,f_t(V)\cap U)$, there is a homotopy through formal Legendrian embeddings to genuine loose Legendrian embeddings.*

Proof. First by Proposition 9.2 there is a homotopy from formal Legendrian embeddings to wrinkled Legendrian embeddings, and the loose chart U is fixed through the homotopy. We denote the wrinkled Legendrians still by $f_t:V\to(W,\xi)$. Suppose there are K embryos in the family of wrinkled Legendrians. Choose K distinct loose charts $(U_i,f_t(V)\cap U_i)\subset(U,f_t(V)\cap U)$.

Now we construct a new Legendrian embedding $g_t:V\to(W,\xi)$. Replace each loose chart by an inside-out wrinkle

$$\bar w(x,u)=(x,\psi_{|x|^2-1}(u)).$$

For each $1\le i\le K$, we define a marking Φ^i_t such that

(1). Φ^i_t is either D^k or $S^{k-1}\times[0,1]$ for any $t\in I^k$;

(2). Φ^i_t is a disk contained in $g_t^{-1}(U_i)$ near $t\in\partial I^k$ so that $\Phi^i_t=\{(x,u)|u=0,|x|\le 1\}$ near the inside-out wrinkle;

(3). $\partial\Phi^i_t$ consists of exactly all the Legendrian embryos other than the wrinkled spheres inside U_i.

By applying Lemma 9.4 one will get a family of smooth Legendrians $\tilde g_t:V\to(W,\xi)$ that is loose. Notice that we have changed f_t for $t\in\partial I^k$. But this is not an issue since $\tilde g_t$ and f_t are smoothly isotopic near $t\in I^k$. By inserting this family of isotopy in between $\{f_t\}_{t\in\partial I^k}$ and $\{\tilde g_t\}_{t\in\partial I^k}$, one will get a family of loose Legendrians that extends $\{f_t\}_{t\in\partial I^k}$.

It suffices to check that $\tilde{g}_t : V \to (W, \xi)$ is formally Legendrian isotopic to $f_t : V \to (W, \xi)$. There are two problems: one is that we have replaced loose charts by inside-out wrinkles, the other is that we have resolved embryos to loose charts. In fact these two procedures are inverse to each other (resolving the inside-out wrinkles will just give us loose charts).

The family $\{g_t\}_{t\in D^k}$ has a loose chart and a number of wrinkles. By resolving the inside-out wrinkles, one can pass to $\{f_t\}_{t\in D^k}$. Since in the front projection a wrinkle is formally isotopic to the zero section, the whole singular part of $\{f_t\}_{t\in D^k}$ is formally isotopic to a loose chart. On the other hand, by resolving the embryos, one can pass to $\{\tilde{g}_t\}_{t\in D^k}$. Again the singular part of $\{\tilde{g}_t\}_{t\in D^k}$ turns out to be formally isotopic to the zero section. This completes the proof. $\square$

In the definition of a loose chart, the condition $\rho > 1$ is necessary. In fact if this condition is not required, then any Legendrian will have such a chart. This can be shown by the h-principle of isocontact embeddings from the standard cube (together with an zig-zag) to the ambient manifold (together with the given Legendrian). However, we need to be a little more careful since there are extra data about the Legendrian being fixed.

This will show that any Legendrian has such a chart for SOME $\rho > 0$. In fact we can show such a chart always exists for ANY $\rho < 1$ by constructing explicitly a zig-zag using Reidmeister move I twice for the zero section.

However, though there are some tricky size issues, there is still a relatively simple criterion of looseness in high dimensions.

Proposition 9.7. *Let Λ be a Legendrian submanifold in $J^1(V)$, and $\pi(\Lambda)$ its front projection in $V \times \mathbb{R}$. If there is a disk $D \subset V \times \mathbb{R}$ being tangent to the $\mathbb{R}$ direction and tranverse to $\pi(\Lambda)$ such that $D \cap \pi(\Lambda)$ is a standard zig-zag. Then Λ is loose.*

Proof. Consider $\pi^{-1}(D \times D_\epsilon^{n-1}) \subset J^1(V)$. Then $\pi^{-1}(D \times D_\epsilon^{n-1}) \simeq J^1(I) \times T^* I_\epsilon^{n-1}$. Now the contactomorphism

$$I^3 \times D_\rho^{2(n-1)} \to J^1(I, I) \times T^* D_\epsilon^{n-1};\ \ (q_1, p_1, z; q, p) \mapsto \left(q_1, p_1, z; \frac{\epsilon}{\rho} q, \frac{\rho}{\epsilon} p\right)$$

shows that one can find a loose chart with required size inside $\pi^{-1}(D \times D_\epsilon^{n-1})$. $\square$